UMHEP-270

Self Consistent Mean Field Approach to Quantum Chromodynamics at Low Momentum

Arthur R. Swift
Department of Physics and Astronomy
University of Massachusetts
Amherst, MA 01003

Abstract

A model of the large spatial distance, zero three-momentum, limit of QCD is developed from the hypothesis that there is an infrared singularity. Single quarks and gluons do not propagate because they have infinite energy after renormalization. The Hamiltonian formulation of the path integral is used to quantize QCD with physical, non-propagating fields. Perturbation theory in the infrared limit is simplified by the absence of self energy insertions and by the suppression of large classes of diagrams due to vanishing propagators. Remaining terms in the perturbation series are re-summed to produce a set of non-linear, renormalizable integral equations which fix both the confining interaction and the physical propagators. Solutions demonstrate self consistency of the concepts of an infrared singularity and non-propagating fields. The Wilson loop is calculated to provide a general proof of confinement. Bethe-Salpeter equations for quark-antiquark pairs and for two gluons have finite energy solutions in the color singlet channel. The choice of gauge is addressed in detail. Large classes of corrections to the model are discussed and shown to support self consistency. Since all elements of QCD are retained, it is possible, in principle, to calculate any hadronic amplitude.

I. Introduction

Although color confinement in Quantum Chromodynamics (QCD) has been established in lattice calculations[1], there are still ambiguities in recovering the continuum limit of the theory. Moreover, realistic lattice calcualtions of physical quantities like bound state masses and weak interaction matrix elements are intrinsically cumbersome. Massive amounts of computation are required and there remain important questions about the best numerical techniques. In addition the results of lattice calculations carry statistical uncertainties. Hence, there is reason to complement the numerical analysis of QCD with analytic investigations of models of the confinement process that are closely connected to the continuum field theory.

This paper presents a model of confinement in QCD. It is derived directly from the equations of the field theory. Altnough quarks and gluons are initially assumed to be confined and a particular interaction is identified as "important" in the large distance limit, the full structure of the theory is retained. The assumptions are justified a posteriori. Residual interactions are calculable with well defined perturbation theory rules. The model developed here differs from others[2] not only in mechanism but also in its reliance on the equations of QCD. One of the attractive features of the model is the absence of colored single particle states. Yet it is possible to construct a Bethe-Salpeter equation for finite energy, color singlet bound states. Even if this model does not embody the "true" confinement mechanism in QCD, it should be useful in the analysis of important theoretical questions such as dynamical symmetry breaking. Moreover, the technique used in constructing a mean field theory in the path integral framework could prove useful in creating wider class of models.

Some of the results have appeared in a series of papers on the connection between the QCD vacuum and confinement.[3-5] The derivation is totally new. This paper is self contained. When there is overlap with earlier work, the discussion is abbreviated.

The model of confinement is based on the observation that physical quarks and physical(= transverse) gluons do not propagate in the non-perturbative vacuum. The momentum space representations of single particle propagators do not have poles. In the Coulomb gauge only the transverse degrees of freedom of the gauge field are quantized. It is the natural gauge in which to explore the consequences of non-propagating quarks and gluons. The problems of the Coulomb gauge are well known[6] and are addressed below. Christ and Lee[7] quantize QCD in non-covariant gauges. The Coulomb gauge Hamiltonian is a function of physical fields. It is used in a path integral to develop a mean field theory for the QCD ground state. Two variations on the conventional path integral formalism are introduced. Counterterms are added and subtracted to define an "unperturbed" Hamiltonian for gluons and quarks propagating in the physical vacuum. The counterterms are fixed by the requirement that there are no quark and gluon self energy insertions in the perturbation series. By hypothesis single particle propagators in the physical vacuum do not have poles at finite energies. Self consistency requires that the calculated counterterms reproduce that property.

The second variation on path integral quantization involves use of the Hamiltonian rather than Lagrangian formulation.[8] Conventionally one integrates out canonically conjugate momentum fields. The Faddeev-Popov determinant[9] appears. Ghost fields are introduced to handle the explicit field dependence of the determinant. In the Coulomb gauge that determinant is known to have zeros. Here the generating function is calculated by coupling external sources to both the gauge fields and their conjugate momenta. The unperturbed Hamiltonian is a quadratic form. One derives propagators not only for particle fields but also for the conjugate momenta. The interaction Hamiltonian is a function of both. The momenta fields play the role of ghosts, but there is no Faddeev-Popov determinant. Although the interaction Hamiltonian is quadratic in momentum fields, it is non-local and has terms of all orders in the ordinary fields. That complication is not a problem since the mean field theory concentrates on a non-perturbative description of the large distance/low momentum limit. Moreover there are anomalous interactions[7] in the Coulomb gauge Hamiltonian which have an equally complicated dependence on the gluon fields.

In the mean field theory the positions of the poles of the single particle propagators depend on an infrared cut-off parameter. As the parameter goes to zero, the poles move to infinity and the propagators vanish. The time component of the gauge field propagates instantaneously in the Coulomb gauge. As the infrared cut-off parameter vanishes, the Coulomb interaction becomes singular and produces confinement. Perturbation theory in the physical vacuum is considerably simplified because all Feynman diagrams with more than one gluon and/or quark in a momentum loop vanish with the infrared cut-off. (The effect is a continuum version of the quenched approximation[10] in lattice calculations.) Loops with a single quark or gluon are infrared finite if integration over the time component of the momentum is performed before removing the cut-off. Exceptions to this rule occur when there are factors of the instantaneous Coulomb interaction to provide compensating singularities. The surviving perturbation theory diagrams are summed to generate a set of equations for the quark and gluon propagators and for the Coulomb interaction. Even though emphasis is on the zero momentum limit, the equations are renormalizable. Moreover, there are solutions consistent with the hypothesized singularity structure. The model is self consistent. Finite energy bound states exist thanks to the effects of vanishing propagators and singular interactions. The Wilson loop[11] has the expected area dependence. In a lowest order calculation, the effective quark-antiquark potential is predicted to be linear in configuration space, at least out to distance where pair production becomes important.

Given the success of the model, it is important to consider possible flaws in its foundations. Clearly the use of the Coulomb gauge is both crucial and the most obvious target of criticism. The choice of gauge is dictated by the need to quantize only physical degrees of freedom. It is precisely those transverse gluon modes which do not propagate as normal particles. In addition, the Coulomb gauge has the virtue that the QCD version of Gauss's law does not need to be imposed as a separate condition on states.[7] On the other hand, the Coulomb gauge is not covariant; the condition $\vec{\nabla}\cdot\vec{A} = 0$ singles out a particular Lorentz frame. Proper quantization in the Coulomb gauge is difficult and leads to interaction terms which are of arbitrarily high order in the coupling constant.[7] There are singularities associated with the vanishing of the Faddeev-Popov determinant which can affect path integral quantization.[12] In addition in any parameter free gauge, it is difficult to verify that any particular result is not an artifact of the choice of gauge.

Lorentz invariance is a problem in any mean field approximation to field theory. If the theory is quantized canonically, it is possible to define boost operators that transform states and operators to an arbitrary Lorentz frmae, at least in principle.[13] In a path integral approach the effects of a boost are equivalent to higher order corrections to the simple states of the effective theory. The choice of frame in the Coulomb gauge is intimately related to both the identification of the physical degrees of freedom of gauge particles and to the choice of the hypersurface of quantization. An arbitrary timelike vector $\eta^\mu [\eta\cdot\eta = \eta^{o2} - \vec{\eta}\cdot\vec{\eta} = 1]$ can be used to define a covariant Coulomb gauge. The theory is quantized on surfaces of constant $\tau = \eta\cdot x$. The gauge condition is $\partial\cdot A - \eta\cdot\partial\ \eta\cdot A = 0$, and $\eta\cdot A$ carries the τ instantaneous interaction. The construction of Christ and Lee[7] can be extended to reproduce a "Hamiltonian" that generates τ translations. In each such covariant Coulomb gauge there is a mean field theory in which $A^\mu - \eta^\mu \eta\cdot A$ does not propagate. In every Lorentz frame there is a confining mean field theory. However, it is not obvious that each such theory is equivalent to the others. The ground states are not identical.

The question of Lorentz invariance can also be approached from the underlying invariance of QCD. All gauges are equally invariant if amplitudes of interest are calculated to all orders in the coupling constant. Since in the present context perturbation theory is simplified by the non-propagation of quarks and gluons, it is possible to calculate the effects of very large classes of diagrams and restore a measure of Lorentz invariance. The suppression of quark and gluon loops reflects the difficulty in creating pairs of infinite energy particles.

Finally it is appropriate to ask whether one should demand explicit Lorentz invariance in a large distance/low momentum theory of confinement. The description of a meson as a quark-antiquark pair connected by a flux tube is certainly not covariant or invariant. If a two-particle bound state is boosted, an infinite number of "sea" particles are created. Thus, any model of hadrons formed from a finite number of constituents necessarily lacks Lorentz invariance, even if it is written in ostensibly co-variant form.

One of the costs of quantization in the Coulomb gauge is the appearance of interactions which have no obvious counterparts in other gauges.[7] Derivation of these anomalous interactions requires a careful transformation from the temporal gauge. Attention must be paid to the Jacobian of the gauge transformation, the integration over group elements which replace unphysical degrees of freedom, and the Weyl ordering[7] of the Hamiltonian. The Jacobian of the transformation turns out to be the Faddeev-Popov determinant. When a square root of the determinant is absorbed into the definition of state vectors, the result is a Hamiltonian which is independent of the Faddeev-Popov determinant. However, new interactions appear which are anomalous in the sense that they do not appear when the theory is quantized directly in the Coulomb gauge.[14] These terms are complicated functions of the gauge fields and the Coulomb Green's function. It has been shown that new interactions are necessary for the cancellation of ultraviolet divergences.[15] However, it appears that their only effect on the zero momentum limit is to produce finite higher order (in coupling constant) corrections.

Singularities due to the vanishing of the Faddeev-Popov determinant do not explicitly appear in the quantization method adopted here. The determinant never appears explicitly. It is buried implicitly in state vectors. However, Christ and Lee are able to recover the correct covariant Feynman rules in the Coulomb gauge while ignoring wave function singularities. Moreover, the determinant vanishes only for sufficiently strong fields.[16] Since the mean field theory is quantized explicitly in the physical vacuum, there is a suppression of the large field components which occur at zero momentum in ordinary perturbation theory. The philosophy of this work is that singularities arise as a result of dynamics. One way to proceed is to use a perturbation theory that is well defined for weak fields, re-sum the series, and then analytically continue. Problems due to the Gribov horizon[12] never explicitly appear.

The above discussion raises the question as to whether the whole calculation is a gauge artifact. There are good reasons for choosing the Coulomb gauge to define a noncovariant mean field theory. Moreover, it would not be surprising if one gauge is favored over all others for the analysis of the zero momentum, large three-dimensional distance limit of QCD in Minkowski space. Moreover, some of the key quantities in this model, i.e. the gluon propagator, are not gauge invariant. Trouble can arise with the choice of gauge when calculations are truncated to finite order in the coupling constant. It is important to be able to calculate large classes of diagrams to all orders in the coupling constant. A complete calculation of a gauge invariant quantity in any gauge is gauge independent. However, even if the self consistency of the model is an artifact of the choice of gauge, these results should still prove useful for the analysis of important theoretical questions. Unlike bag models, potential models, and other models of confinement,[2] the full structure of QCD is retained. Perturbations are unambiguous. For example it is possible to calculate the vacuum expectation values of scalar bound states and the mixing of quark-antiquark states with glueballs.

The mean field approach to path integral quantization is developed in the next section. The starting point is the Hamiltonian in the Coulomb gauge and the result is a set of Feynman rules for diagrams containing quarks, transverse gluons, and timelike (= Coulomb) fields. The gauge group is SU(N). Amplitudes which vanish in the $N\to\infty$ limit, g^2N fixed, are expected to dominate the confinement process.[17] The third section develops the self consistency conditions, most of which have appeared before. The Wilson loop is calculated in the fourth section. Not only does the result display confinement, but the calculation emphasizes the significance of the relationship between non-propagating gluons and an instantaneous, singular Coulomb interaction. Bethe-Salpeter equations for the quark-antiquark bound states and glueball states are derived in section five. The bound state equations have finite energy solutions. The energy spectrum is not calculated in this investigation. The Bethe-Salpeter wave functions are normalized to make possible the calculation of diagrams with propagating color singlet bound states. Anomalous interactions and other miscellaneous points are covered in section six. The final section touches briefly on possible applications and conclusions. The appendices are devoted to important technical details.

II. Quantization in the Coulomb Gauge

The QCD Hamiltonian in the Coulomb gauge is[7]

$$
\begin{aligned}
H = {}& \frac{1}{2}\int d^3r\, B_i{}^a(\vec{r})B_i{}^a(\vec{r}) + \frac{1}{2}\int d^3r\, P_i{}^a(\vec{r})P_i{}^a(\vec{r}) \\
&+ \int d^3r\, \bar{\psi}_\alpha(\vec{r})\left(-i\vec{\gamma}\cdot\vec{\nabla} - g\vec{\gamma}\cdot\vec{A}^a(\vec{r})\,\frac{\lambda^a}{2} + m\right)_{\alpha\beta}\psi_B(\vec{r}) \\
&+ \frac{g^2}{2}\int d^3r\, d^3r'\,\{[f_{abc}\vec{P}^b(\vec{r})\cdot\vec{A}^c(\vec{r}) - \bar{\psi}(\vec{r})\gamma^o\,\frac{\lambda^a}{2}\,\psi(\vec{r})] \qquad (2.1) \\
&\qquad F_{aa'}(\vec{r},\vec{r}')[f_{a'b'c'}\vec{P}^{b'}(\vec{r}')\cdot\vec{A}^{c'}(\vec{r}') - \psi(\vec{r}')\gamma^o\,\frac{\lambda^{a'}}{2}\,\psi(\vec{r}')]\}_W \\
&\qquad + V_1(A) + V_2(A) \quad .
\end{aligned}
$$

Both the gluon field $A_i^a(\vec{r})$ and its conjugate momentum field $P_i^a(\vec{r})$ are divergence free in the Coulomb gauge.

$$\vec{\nabla}\cdot\vec{A}^a = \vec{\nabla}\cdot\vec{P}^a = 0 \quad . \tag{2.2}$$

Implicit in the definition of the quark fields $\Psi_\alpha(\vec{r})$ are color, flavor, and Dirac indices. The quark mass matrix m is diagonal in flavor space. The gauge group is SU(N), and f_{abc} represents the usual set of anti-symmetric structure constants. The NxN Hermitian matrices λ^a satisfy

$$[\lambda^a, \lambda^b] = 2if_{abc}\,\lambda^c \quad , \tag{2.3a}$$

$$tr(\lambda^a \lambda^b) = 2\delta_{ab} \quad . \tag{2.3b}$$

The color magnetic field is

$$B_i^a = \frac{1}{2}\,\varepsilon_{ijk}(\nabla_j A_k^a - \nabla_k A_j^a + gf_{abc}A_j^b \;\; A_k^c) \tag{2.4}$$

To lowest order in the coupling constant g, $F_{aa'}(\vec{r},\vec{r}\,';t)$ is proportional to $\delta_{aa'}/|\vec{r}-\vec{r}\,'|$. Hence, $F_{aa'}(\vec{r},\vec{r}\,';t)$ incorporates the QCD modifications of the ordinary instantaneous Coulomb interaction of quantum electrodynamics. This modified Coulomb potential is defined by[3,7]

$$F_{aa'}(\vec{r},\vec{r}\,';t) = \int d^3r''\; D_{ab}(\vec{r},\vec{r}\,'';t)(-\nabla^2)D_{ba'}(\vec{r}\,'',\vec{r}\,';t) \quad , \tag{2.5}$$

where $D_{ab}(\vec{r},\vec{r}\,'';t)$ is the modified Coulomb Green's function.

$$-[\nabla^2\delta_{ac} + g\,f_{abc}\vec{A}^b(\vec{r})\cdot\vec{\nabla}]D_{cd}(\vec{r},\vec{r}\,';t) = \delta_{ad}\delta^3(r-r') \quad . \tag{2.6}$$

The W subscript around the Coulomb term in the Hamiltonian indicates Weyl ordering of the factors inside the curly brackets. Since $A_i^a(\vec{r})$ and $p_i^a(\vec{r})$ are non-commuting operators, it is necessary to specify their ordering in the quantum Hamiltonian. The final terms in (2.1), $V_1(A)$ and $V_2(A)$, are the anomalous interactions. They are absent when QCD is quantized directly in the Coulomb gauge rather than transformed from a well behaved gauge. Schwinger[18] discovered the need for $V_1(A)$ by requiring that the generators of Lorentz boosts satisfy correct commutation relations. Explicit expressions for $V_1(A)$ and $V_2(A)$ appear in

Appendix A.

Feynman rules in the Coulomb gauge are calculated from the generating function

$$W = \bar{N}\int D(A_i^{\ a})D(P_j^{\ b})D(\bar{\psi}_\alpha)D(\psi_\beta) \exp(i\int dt[\int d^3r(\vec{P}^a\cdot\dot{\vec{A}}^a + i\bar{\psi}\gamma^o\dot{\psi}) - H] \quad , \tag{2.7}$$

where H is augmented by terms coupling fields to external sources. The path integral in (2.7) involves only the physical components of the gluon field. There are no additional gauge constraints in the integrand. Since H is quadratic in momentum fields $P_i^{\ a}(\vec{r})$, it is conventional to explicitly carry out the momentum integrations. The result is the appearance of the Faddeev-Popov determinant and the need for ghost fields.[7] An alternative method of quantization is to treat $A_i^{\ a}(\vec{r})$ and $P_i^{\ a}(\vec{r})$ as equivalent fields. First, however, I switch from configuration space to momentum space.

$$A_i^{\ a}(\vec{r},t) = \int d^4k\, e^{-ik_o t + i\vec{k}\cdot\vec{r}} A_i^{\ a}(k) \quad . \tag{2.8}$$

The momentum field and the quark fields are similarly transformed. The Fourier transform is equivalent to a change of variable in the path integral. The Jacobian of the transformation is a constant that is absorbed into the normalization factor N. S_o is that portion of the action in (2.7) which is linear or quadratic in fields.

$$\begin{aligned} S_o = &- (2\pi)^4 \int d^4k\, d^4p\, \delta^4(p+k) \\ &\{ip_o\, \vec{P}^a(k)\cdot\vec{A}^a(p) + \tfrac{1}{2}\vec{P}^a(k)\cdot\vec{P}^a(p)[1 + F_1(k)] \\ &+ \tfrac{1}{2}\vec{A}^a(k)\cdot\vec{A}^a(p)[\vec{k}^2 + F_2(k)] + \vec{A}^a(k)\cdot\vec{J}^a(p) + \vec{P}^a(k)\cdot\vec{K}^a(p) \\ &- \bar{\psi}(k)\gamma^o p_o\psi(p) + \bar{\psi}(k)[\vec{\gamma}\cdot\vec{p}(1 + G_1(p)) + m(1 + G_2(p))]\psi(p) \\ &+ \bar{\psi}(k)\eta(p) + \bar{\eta}(k)\psi(p)\} \quad . \end{aligned} \tag{2.9}$$

Since both $A_i^{\ a}(k)$ and $P_i^{\ a}(k)$ are transverse, only the transverse components of the external boson sources $J_i^{\ a}(k)$ and $K_i^{\ a}(k)$ are coupled. Thus, $\vec{J}\cdot\vec{A}$ is shorthand for $J_i P_{ij} A_j$ where $P_{ij}(k) = \delta_{ij} - k_i k_j / k^2$. The treatment of quark degrees of freedom is standard; η and $\bar{\eta}$ are quark source terms. Counterterms appear in (2.7). The functions $F_1(k)$ and $F_2(k)$ modify the gluon propagators, and $G_1(k)$ nd $G_2(k)$ enter the quark propagator. These functions are expected to be even, $F_i(k) = F_i(-k)$. The additions to S_o are compensated by the subtraction of identical quadratic terms in the interaction Hamiltonian. The counterterms are fixed by the requirement that S_o describe a mean field theory in which single particle propagators are fully dressed.

The exact generating function is expressed in terms of functional derivatives acting on the free particle generating function.

$$W = \exp(+ iS_I(\frac{i}{(2\pi)^4} \frac{\delta}{\delta J_i{}^a(-k)}, \frac{i}{(2\pi)^4} \frac{\delta}{\delta K_i{}^a(-k)}, \frac{i}{(2\pi)^4} \frac{\delta}{\delta \eta_\alpha(-p)}, \frac{i}{(2\pi)^4} \frac{\delta}{\delta \bar{\eta}_\beta(-p)}) W_o \quad , \tag{2.10}$$

with

$$\begin{aligned} W_o &= \bar{N} \int D(A_i{}^a) D(P_i^a) D(\psi) D(\bar{\psi}) e^{iS_o} \\ &= \bar{N}' \exp(-i(2\pi)^4 \int d^4k\, d^4p\, \delta^4(p+k) \\ &\quad [\frac{1}{2} \frac{1+F_1(k)}{d(k)} \vec{J}^a(k) \cdot \vec{J}^a(p) + \frac{1}{2} \frac{\vec{k}^2 + F_2(k)}{d(k)} \vec{K}^a(k) \cdot \vec{K}^a(p) \\ &\quad + \frac{ik_o}{d(k)} \vec{K}^a(k) \cdot \vec{J}^a(p)]) \, x W_F(\eta, \bar{\eta}) \quad , \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} d(k) &= k_o{}^2 - (k^2 + F_2(k))(1 + F_1(k)) + i\varepsilon \\ &= k_o{}^2 - \omega(k)^2 + i\varepsilon \quad . \end{aligned} \tag{2.12}$$

The fermionic generating function is

$$W_F = \exp(-i(2\pi)^4 \int d^4k\, d^4p\, \delta^4(p+k) \frac{\bar{\eta}(k)[\gamma^o p_o - \vec{\gamma} \cdot \vec{p}(1 + G_1(p)) + m(1 + G_2(p))]\eta(p)}{d_F(p)}) \tag{2.13}$$

where

$$d_F(p) = p_o^{\,2} - \vec{p}^{\,2}(1 + G_1(p))^2 - m^2(1 + G_2(p))^2 \quad . \tag{2.14}$$

The standard path integral formalism yields the following "exact" single gluon propagators:

$$<A_i^{\,a}(k)\ A_j^{\,b}(p)> = \frac{i}{(2\pi)^4}\ \delta_{ab}\delta^4(p+k)\ P_{ij}(\vec{k})\ \frac{1 + F_1(k)}{d(k)} \quad , \tag{2.15a}$$

$$<P_i^{\,a}(k)\ P_j^{\,b}(p)> = \frac{i}{(2\pi)^4}\ \delta_{ab}\delta^4(p+k)\ P_{ij}(\vec{k})\ \frac{k^2 + F_2(k)}{d(k)} \, , \tag{2.15b}$$

$$<P_i^{\,a}(k)\ A_j^{\,b}(p)> = \frac{i}{(2\pi)^4}\ \delta_{ab}\delta^4(p+k) P_{ij}(\vec{k})\ \frac{ik_o}{d(k)} \quad . \tag{2.15c}$$

The transverse projection operators arise from the fact that only the transverse components of the vector sources occur in (2.11). The quark propagator is

$$<\psi_\alpha(p)\bar{\psi}_\beta(k)> = \frac{i}{(2\pi)^4}\ \delta^4(p+k)\ \frac{[-\gamma^o k_o + \vec{\gamma}\cdot\vec{k}(1 + G_1(k)) + m(1 + G_2(k))]_{\alpha\beta}}{d_F(k)} \tag{2.16}$$

The fundamental hypothesis of the model is that the counterterm functions in (2.15) and (2.16) develop infrared divergences.

Interactions are described by $S_I = \int H_I(A,P,\Psi,\bar{\Psi})dt$. In momentum space this becomes

$$\begin{aligned}
S_I(A,P,\psi,\bar{\psi}) &= (2\pi)^4 \int d^4p\ d^4k\,\delta^4(p+k)[\tfrac{1}{2}\ \vec{P}^a(p)\cdot\vec{P}^a(k)\ F_1(k) \\
&+ \tfrac{1}{2}\ \vec{A}^a(p)\cdot\vec{A}^a(k)\ F_2(k) + \bar{\psi}(p)[\vec{\gamma}\cdot\vec{k}\ G_2(k) + mG_2(k)]\psi(k)] \\
&- ig(2\pi)^4\ f_{abc}\int d^4 1 d^4 2 d^4 3\ \delta^4(1+2+3)\ \vec{A}^a(1)\cdot\vec{A}^c(3)\ \vec{1}\cdot\vec{A}^b(2) \\
&- \frac{g^2}{4}\ (2\pi)^4\ f_{abc}f_{ade}\int d^4 1 d^4 2 d^4 3 d^4 4\ \delta^4(1+2+3+4)\vec{A}^b(1)\cdot\vec{A}^d(3)\ \vec{A}^c(2)\cdot\vec{A}^e(4) \\
&+ \frac{g}{2}\ (2\pi)^4 \int d^4 1 d^4 2 d^4 3\ \bar{\psi}(1)\vec{\gamma}\cdot\vec{A}^a(3)\ \lambda^a\psi(2)\ \delta^4(1+2+3) - \int dt(V_1(A) + V_2(A)) \\
&- \frac{g^2}{2}\ (2\pi)^7 \int d^4 1 d^4 2 d^4 3 d^4 4 dx\ \delta(1_o + 2_o + 3_o + 4_o + x) \\
&\qquad [f_{abc}\vec{P}^b(1)\cdot\vec{A}^c(2) - \tfrac{1}{2}\ \bar{\psi}(1)\gamma^o\lambda^a\psi(2)]\ F_{aa'}(-\vec{1}-\vec{2},-\vec{3}-\vec{4};x) \\
&\qquad [f_{a'de}\ \vec{P}^d(3)\cdot\vec{A}^e(4) - \tfrac{1}{2}\ \bar{\psi}(3)\gamma^o\lambda^{a'}\psi(4)]
\end{aligned} \tag{2.17}$$

Explicit expressions for the anomalous interaction terms are given in Appendix A. Except for the restriction to spatial components, the three gluon, four gluon, and quark-quark-gluon terms are standard QCD interactions. The final term in (2.17) is the modified Coulomb interaction that results when the time component of the gauge field is eliminated by Gauss's Law. The operator transmitting the color force has the Fourier transform

$$F_{aa'}(\vec{r},\vec{r}\,';t) = \int d^3p\,d^3k\;dx\;e^{-ixt}e^{i\vec{p}\cdot\vec{r}}e^{i\vec{k}\cdot\vec{r}\,'}F_{aa'}(\vec{p},\vec{k};x) \quad . \tag{2.18}$$

Ultimately it will be necessary to calculate this function from its definition in terms of the modified Green's function $D_{ab}(\vec{r},\vec{r}\,';t)$. If the Green's function has a Fourier transform like (2.18), then

$$F_{aa'}(\vec{p},\vec{k};x) = (2\pi)^3\int d^3s\;dy\,dz\;D_{ab}(\vec{p},\vec{s};y)s^2\;D_{ba'}(-\vec{s},\vec{k};z)\,\delta(x-y-z) \quad . \tag{2.19}$$

The differential equation (2.6) becomes an integral equation in momentum space.

$$\begin{aligned} D_{ab}(\vec{p},\vec{k};x) &= \frac{\delta_{ab}\delta^3(\vec{p}+\vec{k})\,\delta(x)}{(2\pi)^3\;\vec{p}^{\,2}} \\ &\quad + ig\;f_{ace}\,\frac{1}{\vec{p}^{\,2}}\int d^4s\,\vec{A}^c(s)\cdot\vec{p}\;D_{eb}(\vec{p}-\vec{s},\vec{k};x-s_o) \quad . \end{aligned} \tag{2.20}$$

If (2.20) is expanded in a perturbation series and the result is inserted in (2.19), one can prove that

$$F_{ab}(\vec{p},\vec{k};x) = \frac{d}{dg}\,[gD_{ab}(\vec{p},\vec{k};x)] \quad . \tag{2.21}$$

In the g=0 limit $F_{ab}(\vec{p},\vec{k};x)$ is equal to the first term in (2.20). More generally the Coulomb part of S_I, as well as V_1 and V_2, contains interactions of all orders in g and can lead to the creation of an arbitrarily large number of gluons.

The mean field theory of confinement is defined by the following set of assumptions:

1. There is an infrared singularity in the theory that can be controlled by a cut-off parameter μ.

2. The counterterm functions F_1, F_2, G_1, G_2 diverge as $\mu \to 0$. Each function has a term proportional to a constant $\lambda(\mu)$, where

$\lambda(\mu) \to \infty$ as $\mu \to 0$.

3. In Feynman diagrams integrations over p_o, the time component of a loop momentum, are to be performed before setting $\mu = 0$.

4. The vacuum expectation value (VEV) of the modified Coulomb interaction has the form

$$\langle F_{ab}(\vec{p},\vec{k};x)\rangle = \frac{\delta_{ab}\delta^3(\vec{p}+\vec{k})\delta(x)}{(2\pi)^3} F(\vec{p}) \quad . \tag{2.22}$$

As $\mu \to 0$, the function $F(\vec{p})$ develops a singularity at $\vec{p} = 0$. The singularity is of the form $F(\vec{p}) = \lambda(\mu)\delta^3(\vec{p}) + \bar{F}(\vec{p})$, and $\bar{F}(\vec{p})$ is not singular at $\vec{p} = 0$.

These assumptions are shown to be self consistent in the next section.

Appendix B contains a discussion of the complicated multi-gluon dynamics buried in the deceptively simple notation for the Coulomb term in (2.17). Those interactions are best computed in terms of an operator product expansion for $F_{ab}(\vec{p},\vec{k};x)$. The vacuum expectation values of $F_{ab}(\vec{p},\vec{k};x)$ and $D_{ab}(\vec{p},\vec{k};k)$ play a prominent role as the effective propagators for the Coulomb interaction.

Hypotheses 1 and 2 imply that at fixed momentum all single particle propagators vanish in the $\mu = 0$ limit. For example the <AA> propagator for gluons becomes proportional to $F_1/(F_1F_2) \to \lambda^{-1} \to 0$ when $F_1 \gg 1$ and $F_2 \gg k^2$. Moreover, the pole in the k_o plane moves to $k_o = \pm(F_1F_2)^{1/2} \to \pm\lambda \to \pm\infty$ as $\mu \to 0$. If a propagator occurs inside a momentum loop, there are, according to 3, contributions from encircling the k_o poles. The residue of the pole of an <AA> propagator is proportional to $F_1/(F_1F_2)^{1/2} \simeq \lambda^o$. The residue is finite at $\mu = 0$. Thus, a momentum loop with a single quark or gluon line (and a number of instantaneous Coulomb lines) has a finite infrared limit. In general each momentum integration encircles a set of poles and compensates for one factor of λ^{-1}. Quark lines are treated in the same way. Diagrams with two or more gluons and/or quarks in a momentum loop vanish in the infrared limit. For n particles, a momentum loop is of order λ^{1-n}. In addition, the interactions in (2.17) never produce a momentum loop composed entirely of Coulomb lines. As a result there are no spurious divergences from k_0 integrations.

Assumption 4 implies the existence of an infrared singular propagator for Coulomb interactions. The singularity produces a positive power of the divergent parameter λ. Hence, finite amplitudes can occur when vanishing propagators are matched by either momentum integrations or singular Coulomb propagators. If the Feynman diagram for a particular process contains L momentum loops, G gluon propagators (of any type), Q quark propagators, and C singular Coulomb propagators, the amplitude will be proportional to λ^M where

$$M = L + C - G - Q \quad . \tag{2.23}$$

In Appendix C the correlations between L, G, Q, C, and the number of vertices of various types are used to show that

$$M = 1 - n_{gggg} - n_{qqg} - n_{ggg} \quad . \tag{2.24}$$

The maximum value of M is 1, and it occurs when there are no four gluon vertices ($n_{gggg} = 0$), three gluon vertices ($n_{ggg} = 0$), or quark-quark-gluon vertices ($n_{qqg} = 0$). Diagrams with $M < 0$ vanish. An example of an $M=1$ amplitude is the set of ladder diagrams for quark-antiquark scattering via singular Coulomb exchange. (There is a constraint condition that eliminates the $M=1$ term to leave a finite function.) Diagrams with $M=1$ are responsible for the infrared divergence in the counterterm functions. It is important to remember that there are no quark or gluon self energy diagrams in the perturbation series. The mean field condition requires exact cancellation against the counterterms. The operator expansion for the Coulomb interaction constitutes a similar "exact" treatment of self energy insertions in a Coulomb line. The absence of self energy effects coupled with the vanishing of large classes of multi-particle diagrams leads to a much simplified perturbation series.

There is a technical point that must be mentioned. One could imagine a complicated diagram with a number of particle propagators. Contour integrals are evaluated by the residue theorem. It is possible that in a particular loop setting $k_o = \pm\lambda$ + finite terms at one particular pole could lead to a cancellation of λ dependence in the denominators of other propagators. In that case the naive arguments on counting powers of λ would be wrong. Explicit calculation of a representative set of diagrams shows that this cancellation does not occur.

In constructing a set of Feynman rules, one must recognize that there are three spearate gluon propagators corresponding to <AA>, <PP>, and <AP>. There are two Coulomb propagators which are distinguished by their behavior at $\vec{p} = 0$. The VEV of the modified Coulomb interaction is singular in that limit, while the second propagator is the non-singular VEV of the Green's function. The Green's function propagator is needed to describe the multi-gluon interactions inherent in the Coulomb, V_1, and V_2 contributions to S_I.

III. Self Consistent Mean Field Theory

The amplitudes corresponding to Feynman diagrams depend on six unknown functions. The mean field model makes sense only if these functions can be calculated non-perturbatively. The simplest function to analyze is the VEV of the modified Green's function. Given the generating function W in (2.10), one can calculate the VEV of any function of field operators. The perturbation expansion of $D_{ab}(\vec{p},\vec{k};x)$ is given in (B1). The VEV of D_{ab} has the diagrammatic representation of Figure 1. Diagrams which vanish in the $\lambda \to \infty$ limit have been eliminated. A refinement of the argument of Appendix C shows that terms in the expansion of $\langle D_{ab}\rangle$ are proportional to λ^M with $M = -n_{ggg} - n_{gggg} - n_{qqg}$. (The presence of two external Coulomb lines reduces (2.24) by one.) The only nonvanishing diagrams are those in which physical gluons are emitted and re-absorbed by the gluon line. The perturbation series for this sub-set can be re-summed to produce a Dyson equation for $\langle D_{ab}\rangle$.[4]

$$D(\vec{k}) = \frac{1}{\vec{k}^2}\,\frac{1}{1 - gI(\vec{k})} \quad , \tag{3.1}$$

where $D(\vec{k})$ is defined in (B3). If the DgD_o vertex function in Figure 1 ($D_o = 1/k^2$) is replaced by its zeroth order value,

$$I(\vec{k}) = \frac{N}{2(2\pi)^3} \int d^3p \, \frac{p^2k^2 - (\vec{p}\cdot\vec{k})^2}{p^2k^2} \, A(\vec{p}) g D(\vec{p}-\vec{k}) \quad , \tag{3.2}$$

for the gauge group SU(N). The role of vertex corrections and other higher order corrections is addressed in Appendix E. The propagator function $A(\vec{p})$ is of order λ^o.[5]

$$\begin{aligned} A(\vec{p}) &= \frac{i}{\pi} \int_{-\infty}^{\infty} dp_o \, \frac{1 + F_1(\vec{p})}{p_o^2 - (1 + F_1(\vec{p}))(p^2 + F_2(\vec{p}))} \\ &= \left(\frac{1 + F_1(\vec{p})}{p^2 + F_2(\vec{p})}\right)^{1/2} \quad . \end{aligned} \tag{3.3}$$

I have used the fact, to be verified shortly, that $F_1(\vec{p})$ and $F_2(\vec{p})$ are independent of p_o in the $\lambda \to \infty$ limit.

Equations (3.1) and (3.2) were derived in an earlier work on the relationship between confinement and properties of the QCD vacuum.[4] The extensive discussion will not be repeated here. Asymptotic freedom predicts that, within logarithms, $A(\vec{s}) \to 1/s$ and $D(\vec{s}) \to 1/s^2$ as $\vec{s} \to \infty$. Renormalization is necessary to remove a logarithmic divergence in (3.2). Analysis of the $\vec{p} \to 0$ limit shows that if $A(\vec{p})$ approaches a constant, $D(\vec{p})$ is proportional to $(p^2)^{-5/4}$. Conventional perturbation theory sets $A(\vec{p}) = 1/p$ and $D(p) = 1/p^2$ in lowest order.

The $\vec{p} = 0$ enhancement of $D(\vec{p})$ is promoted to a true infrared singularity in $F(\vec{p}) = d[gD(\vec{p})]/dg$. Differentiation of (3.1) leads to

$$F(\vec{p}) = \frac{1}{p^2} \, \frac{1 + g^2 J(\vec{p})}{[1 - gI(\vec{p})]^2} \tag{3.4}$$

with

$$J(\vec{p}) = \frac{N}{2(2\pi)^3} \int d^3k \, \frac{[p^2k^2 - (\vec{p}\cdot\vec{k})^2]}{p^2k^2} \, A(\vec{k}) \, F(\vec{p}-\vec{k}) \quad . \tag{3.5}$$

These equations also appeared in the paper on the QCD vacuum.[4] Since $F(\vec{p}) \to 1/p^2$ as $p \to \infty$, the integral defining $J(\vec{p})$ is logarithmically divergent. After renormalization, the solution of (3.4) should lead to an infrared singularity in $F(\vec{p})$.

$F(\vec{p})$ is infrared singular if, as $\vec{p} \to 0$, $F(\vec{p}) \propto (p^2)^{-n}$ and $n > 3/2$. When n is in the range $3/2 < n < 5/2$, the singularity can be controlled by a simple subtraction procedure. If

$$F(\vec{p}) = \frac{f}{[p^2 + \mu^2]^n} \quad , \tag{3.6}$$

then for an arbitrary function $H(\vec{k})$

$$I(\vec{p}) = \int d^3k \, F(\vec{k}-\vec{p}) H(\vec{k}) = \lambda(\mu) H(\vec{p}) + \bar{I}(\vec{p}) \quad . \tag{3.7}$$

The integral $\bar{I}(p)$ is finite as $\mu \to 0$ and

$$\lambda(\mu) = \frac{2\pi \, \Gamma(3/2) \Gamma(n-3/2)}{\mu^{2n-3} \, \Gamma(n)} . \tag{3.8}$$

Equation (3.7) is equivalent to writing

$$F(\vec{p}) = \lambda(\mu) \, \delta^3(p) + \bar{F}(\vec{p}) \quad . \tag{3.9}$$

Integrals containing $\bar{F}(\vec{p})$ are defined from (3.6) with the stipulation that μ dependent divergences are to be discarded. For example, in (3.5) $F(\vec{p}-\vec{k})$ can be replaced by $\bar{F}(\vec{p}-\vec{k})$, since $p^2k^2 - (\vec{p}\cdot\vec{k})^2$ vanishes at $\vec{p} = \vec{k}$. The integral is finite at $\mu = 0$.

The mean field theory is complete when $A(\vec{p})$ and the quark counterterm functions are calculated. Equation (2.15) for the gluon propagator is assumed to be exact. The sum of self energy insertions in a gluon line must vanish. Order g^2 corrections to gluon and quark lines are shown in Figure 2. Counterterm insertions are indicated by crosses. According to (2.24) only the diagrams with a singular Coulomb line contribute in the $\lambda \to \infty$ limit. The other self energy diagrams are of order λ^{-1}. The second order correction to (2.15a) is

$$\begin{aligned} \langle A_i^{\,a}(\vec{k}) \, A_j^{\,b}(\vec{p}) \rangle_{g^2} &= \frac{i}{(2\pi)^4} \, \delta_{ab} \delta^4(\vec{p}+\vec{k}) P_{ij}(\vec{k}) \\ &\frac{1}{d(k)} \Big\{ -k_o^{\,2} [F_1(\vec{k}) - \frac{g^2 N}{4(2\pi)^3} \int d^3s \, F(\vec{s}-\vec{k}) A(\vec{s}) \; tr[P(\vec{s}) \; P(\vec{k})]] \\ &- (1 + F_1(\vec{k}))^2 [F_2(\vec{k}) - \frac{g^2 N}{4(2\pi)^3} \int d^3s \, \frac{F(\vec{s}-\vec{k})}{A(\vec{s})} \; tr[P(\vec{s}) \; P(\vec{k})]] \Big\} \frac{1}{d(k)} \quad . \end{aligned} \tag{3.10}$$

The identical combination of F_1, F_2, and integrals occurs in <PP> and <PA>. Hence, the condition that self energy corrections vanish fixes F_1 and F_2 (to order g^2).

$$F_1(\vec{k}) = \frac{\alpha}{2} \int d^3s\, F(\vec{s}-\vec{k}) A(\vec{s}) \quad \text{tr}\, [P(\vec{s})P(\vec{k})] \quad , \tag{3.11a}$$

$$F_2(\vec{k}) = \frac{\alpha}{2} \int d^3s\, F(\vec{s}-\vec{k}) \frac{1}{A(\vec{s})} \quad \text{tr}\, [P(\vec{s})P(\vec{k})] \quad , \tag{3.11b}$$

with $\alpha = g^2N/(2(2\pi)^3)$. The corresponding quark calculation for each flavor leads to

$$G_1(\vec{k}) = \frac{\alpha'}{2} \int d^3s\, F(\vec{s}-\vec{k}) \frac{\vec{k}\cdot\vec{s}}{k^2} \frac{(1 + G_1(\vec{s}))}{E(\vec{s})} \quad , \tag{3.12a}$$

$$G_2(\vec{k}) = \frac{\alpha'}{2} \int d^3s\, F(\vec{s}-\vec{k}) \frac{(1 + G_2(\vec{s}))}{E(\vec{s})} \quad , \tag{3.12b}$$

where $\alpha' = [(N^2-1)/N^2]\alpha$ and

$$E(\vec{s})^2 = \vec{s}^{\,2}(1 + G_1(\vec{s}))^2 + m^2(1 + G_2(\vec{s}))^2 \quad . \tag{3.13}$$

According to (3.11) and (3.12) the counterterm functions are functions of three-momentum only; there is no dependence on k_o.

The mean field equations have a remarkable property in the infrared limit. When (3.9) is used for $F(\vec{s}-\vec{k})$, the gluon functions become

$$F_1(\vec{k}) = \lambda\alpha A(\vec{k}) + \bar{F}_1(\vec{k}) \quad , \tag{3.14a}$$

$$F_2(\vec{k}) = \lambda\alpha/A(\vec{k}) + \bar{F}_2(\vec{k}) \quad , \tag{3.14b}$$

where the bar over a function indicates the absence of an infrared singularity. (i.e. $F(\vec{k})$ is replaced by $\bar{F}(\vec{k})$.) Assumption 2 of the model and (3.14) are mutually consistent since $A(\vec{k})$ is infrared finite.[5]

$$A(\vec{k})^2 = \frac{1 + F_1(\vec{k})}{k^2 + F_2(\vec{k})} = \frac{\alpha\lambda A(\vec{k}) + 1 + \bar{F}_1(\vec{k})}{\frac{\alpha\lambda}{A(\vec{k})} + k^2 + \bar{F}_2(\vec{k})}$$

$$= \frac{1 + \bar{F}_1(\vec{k})}{k^2 + \bar{F}_2(\vec{k})} \quad . \tag{3.15}$$

The poles of the gluon propagator are located at

$$\begin{aligned} k_o &= \pm\, [(1 + F_1(\vec{k}))(k^2 + F_2(\vec{k}))]^{1/2} \\ &= \pm\, \{\alpha\lambda + [(1 + \bar{F}_1(\vec{k}))(k^2 + \bar{F}_2(\vec{k}))]^{1/2}\}. \end{aligned} \tag{3.16}$$

As $\lambda \to \infty$ the poles move to $k_o = \pm\infty$.

The infrared singularity in the quark functions can also be isolated

$$\begin{aligned} 1 + G_i(\vec{k}) &= 1 + \frac{\alpha'\lambda}{2E(\vec{k})}\,(1 + G_i(\vec{k})) + \bar{G}_i(\vec{k}) \\ &= \frac{1 + \bar{G}_i(\vec{k})}{1 - \dfrac{\alpha'\lambda}{2E(\vec{k})}} = \left(1 + \frac{\alpha'\lambda}{2\bar{E}(\vec{k})}\right)(1 + \bar{G}_i(\vec{k})) \quad , \end{aligned} \tag{3.17}$$

where

$$E(\vec{k}) = \frac{[k^2(1 + \bar{G}_1)^2 + m^2(1 + \bar{G}_2)^2]^{1/2}}{1 - \dfrac{\alpha'\lambda}{2E(\vec{k})}} = \frac{\bar{E}(\vec{k})}{1 - \dfrac{\alpha'\lambda}{2E(\vec{k})}} = \frac{\alpha'\lambda}{2} + \bar{E}(\vec{k}) \quad . \tag{3.18}$$

Using (3.18) in the integrals defining G_1 and G_2, one finds

$$\bar{G}_1(\vec{k}) = \frac{\alpha}{2} \int d^3s\; \bar{F}(\vec{s}-\vec{k})\, \frac{1 + \bar{G}_1(\vec{s})}{\bar{E}(\vec{s})}\, \frac{\vec{k}\cdot\vec{s}}{k^2} \quad , \tag{3.19a}$$

$$\bar{G}_2(\vec{k}) = \frac{\alpha'}{2} \int d^3s\; \bar{F}(\vec{s}-\vec{k})\, \frac{1 + \bar{G}_2(\vec{s})}{\bar{E}(\vec{s})} \quad . \tag{3.19b}$$

Again there is no dependence on the infrared cut-off parameter. The poles of the quark propagator are located at

$$k_o = \pm\, \{\frac{\alpha'\lambda}{2} + \bar{E}(\vec{k})\} \quad . \tag{3.20}$$

As the cut-off parameter $\mu \to 0$, the poles move to $k_o = \pm\infty$.

In order to fully validate the mean field theory model, it is necessary to show that there exist solutions to the propagator equations with the expected behavior. The gluon equations must be solved **simultaneously** for $D(\vec{p})$, $F(\vec{p})$, and $A(\vec{p})$. Then $F(\vec{p})$ is used to determine $G_1(\vec{p})$ and $G_2(\vec{p})$. First, however, ultraviolet divergences must be removed. The renormalization of the pairs (3.1), (3.2) and (3.4), (3.5) was discussed in reference 4. A set of renormalization constants are defined by $D(\vec{k}) = Z_D D_R(\vec{k})$, $A(\vec{k}) = Z_A A_R(\vec{k})$, $g = Z_g g_R$, and $F(\vec{k}) = Z_F F_R(\vec{k})$.

In addition $D_R(\vec{k})$ is replaced by a running coupling constant.

$$g_R D_R(\vec{k}) = g(\vec{k})/k^2 \quad . \tag{3.21}$$

The renormalized equations with a subtraction at $k^2 = \nu^2$ are

$$\frac{1}{g(k)} = \frac{1}{g(\nu)} - [I_R(\vec{k}) - I_R(\nu)] \quad , \tag{3.22}$$

where $I_R(\vec{k})$ is given by (26) with the replacements $A(\vec{p}) \to A_R(\vec{p})$, $gD(\vec{p}-\vec{k})$ $\to g(\vec{p}-\vec{k})/(\vec{p}-\vec{k})^2$. A second fundamental equation is

$$F_R(\vec{k}) = \frac{1}{k^2} \frac{g(k)^2}{g(\nu)^2} [\nu^2 F_R(\vec{\nu}) + g(\nu)^2 [J_R(\vec{k}) - J_R(\vec{\nu}))] \tag{3.23}$$

with $J_R(\vec{p})$ equal to (3.5) with $A(\vec{p}) \to A_R(\vec{p})$ and $F(\vec{p}-\vec{k})$ $F_R(\vec{p}-\vec{k})$.

Renormalization of the equation for $A(\vec{p})$ is complicated by the fact that the integral defining $\bar{F}_2(\vec{k})$ is quadractically divergent, and two subtractions are necessary. One divergence is absorbed into Z_A, and the other is fixed by the value of $A_R(\vec{k})$ at some reference momentum p.

$$\begin{aligned} &\frac{1 + \bar{F}_{1R}(k) - \bar{F}_{1R}(\vec{\nu})}{A_R(\vec{k})^2} - \frac{1 + \bar{F}_{1R}(\vec{p}) - \bar{F}_{1R}(\vec{\nu})}{A_R(\vec{p})^2} \\ &= k^2 - p^2 + \bar{F}_{2R}(\vec{k}) - \bar{F}_{2R}(\vec{p}) - \frac{k^2 - p^2}{\nu^2 - p^2} [F_{2R}(\vec{\nu}) - F_{2R}(\vec{p})] \quad . \end{aligned} \tag{3.24}$$

The derivation of (3.24), together with constraints on the renormalization constants, is given in Appendix D. $\bar{F}_{1R}$ and $\bar{F}_{2R}$ are given by (3.11) with all functions replaced by their renormalized values. The renormalized coupling constant implicit in the definition of $\bar{F}_{iR}$ is equal to the running coupling constant evaluated at $k^2 = \nu^2$.

The mean field theory model is consistent with its fundamental hypotheses if $F_R(\vec{k})$ is infrared singular. The analysis of references 4 and 5 suggests that as $\vec{k} \to 0$, a possible solution has $g(k) \to \infty$ and $A_R(\vec{k}) \to$ constant. I define $A_R(0) = 1/m$. The scale of the theory is set by m, an arbitrary parameter with dimensions of mass. Using this ansatz I find the following set of equations:

$$\frac{1}{g(k)} = -\,[I(k) - I(0)] \quad , \tag{3.25a}$$

$$F(\vec{k}) = \frac{1}{k^2}\frac{g(k)^2}{g(\nu)^2}\,[1 + g(\nu)^2(J(k) - J(\nu))] \quad , \tag{3.25b}$$

$$\frac{1 + \bar{F}_1(k) - \bar{F}_1(\nu)}{A(k)^2} - \frac{1 + \bar{F}_1(0) - \bar{F}_1(\nu)}{A(0)^2} = k^2 + \bar{F}_2(k) - \bar{F}_2(0) - \frac{k^2}{\nu^2}\,[\bar{F}_2(\nu) - \bar{F}_2(0)] \quad . \tag{3.25c}$$

Since the coupling constant and all functions are renormalized, the R subscript has been dropped. The linear nature of the integral equation for $F(\vec{k})$ is used to set $\nu^2 F(\vec{\nu}) = 1$. Since ν is an arbitrary subtraction point, solutions to (3.25) should not be sensitive to its value. The limit $A(0) \neq 0$ has been imposed on (3.25). However, solutions will not be stable under iteration if that choice is not consistent with the equations.

One can show analytically that all functions approach their asymptotically free values in the ultraviolet limit. There are logarithmic modifications to power law behavior, and the coefficient of the logarithms differs from that predicted by conventional QCD calculations.[4] The difference is expected, since the ordering of infrared and ultraviolet limits eliminates diagrams which contribute to standard perturbative QCD calculations. It is important to note, however, that (3.25) is fully renormalized without the missing diagrams.

Analytic solutions to (3.25) are available in the $\vec{k} \to 0$ limit.

$$\lim_{\vec{k}\to 0} g(k) = \left(\frac{21\pi}{2N}\right)^{1/2} (k/m)^{-1/2} \quad , \tag{3.26a}$$

$$\lim_{\vec{k}\to 0} A(\vec{k}) = 1/m \quad , \tag{3.26b}$$

$$\lim_{\vec{k}\to 0} F(\vec{k}) = \frac{f}{k^2}\left(\frac{k^2}{m^2}\right)^{-1\pm i\theta} \tag{3.26c}$$

where $\theta = 0.0847$ is the solution of

$$\frac{64}{21\pi} = \frac{\sin\pi\theta}{\pi\theta}\,\frac{1}{(1+\theta^2)\cosh\pi\theta} \quad , \tag{3.27}$$

and f is not fixed in the infrared limit. If $64/21\pi = 0.97 \simeq 1$, $\theta \simeq 0$. The presence of an complex power in (3.26c) is a defect of this lowest order calculation. A linear, non-relativistic potential in configuration space corresponds to $\theta = 0$ and $F(k) \propto k^{-4}$. Fortunately θ is small, and there is the possibility that higher order correction could produce a purely real power law dependence for $F(\vec{k})$. In the non-relativistic limit, the Fourier transform of $F(\vec{k})$ is the configuration space potential. Since m is the dimensionful scale factor in the potential, the size of a color singlet bound state is of order 1/m. The deviation from a linear potential for this small value of θ does not become significant (i.e. 10%) until $r \simeq 400/m$. Pair production of hadrons becomes important well before that limit. This argument also suggests the nature of important corrections to $F(\vec{k})$. Appendix E is devoted to a discussion of corrections, except those involving bound states. The general topic of bound states is covered in section 5.

Appendix F presents an alternate calculation of $F(\vec{k})$. Direct evaluation of the VEV of the integral defining F_{ab} as the product of two Green's function operators leads to exactly the same singular behavior. One interprets the result as due to the formation of a "bound state" of two D type Coulomb lines interacting by gluon exchange.

The quark equations are renormalized in Appendix D. The result is

$$1 + \bar{G}_i(k) = 1 + [I\bar{G}_i(k) - I\bar{G}_i(\nu)] \quad , \tag{3.28}$$

where $I\vec{G}_i(\vec{k})$ is the integral on the right hand side of (3.19). The quark mass and all functions are renormalized. The renormalized coupling constant α'_R is assumed to satisfy $\alpha'_R/\alpha_R = \alpha'/\alpha$. (See Appendix D.)

A numerical solution of the complete set of equations is shown in Figure 3. Since there exist in lowest order solutions that fulfill the self consistency conditions, the next step is to prove confinement and the formation of finite energy, color singlet bound states.

IV. Wilson Loop

The mean field model is designed to confine color. One way to prove confinement is to compute the VEV of the Wilson Loop for imaginary time τ.[11,19] The calculation is a non-trivial test of the model. The result depends both on the instantaneous nature of the Coulomb interaction and on the impossibility of propagating physical gluons over finite time intervals. The quantity of interest is

$$L \equiv \langle P(e^{i\oint A\cdot d\ell})\rangle , \tag{4.1}$$

where P indicates ordering of field operators around a closed loop. The loop is fixed to be a rectangle with corners (0,0), $(\vec{n}R,0)$, $(\vec{n}R,T)$, and (0,T). With this choice, L is equal to

$$\begin{aligned} L = \langle tr\{P\exp(i\int_o^R d\sigma\, A_n(\vec{n}\sigma,o))\; P\exp(i\int_o^T d\tau\, A_4(\vec{n}R,\tau)) \\ P\exp(i\int_R^o d\sigma\, A_n(\vec{n}\sigma,\tau))\; P\exp(i\int_T^o d\tau(o,\tau))\}\rangle \quad . \end{aligned} \tag{4.2}$$

In (4.2) $A_\mu(\vec{x},\tau) = A_\mu{}^a(\vec{x},\tau)\lambda^a$ and $A_n = \vec{n}\cdot\vec{A}$. Before (4.2) can be evaluated, it is necessary to define a propagator for the time component of the gluon field. In the Coulomb gauge $A_o{}^a(\vec{x},t)$ is eliminated from the Hamiltonian. However, the coupling of the time component of the quark current to the modified Coulomb interaction shows how an external source coupled to $A_o{}^a(\vec{x},t)$ would enter the Hamiltonian. The free particle action in (2.11) is changed to

$$W_o' = W_o \exp[-i(2\pi)^4 \int d^4p\, d^4k\,\delta^4(p+k) J_o{}^a(k)\, \frac{g^2}{2}\, F(\vec{k})\, J_o{}^a(p)] \quad . \tag{4.3}$$

The corresponding propagator is

$$\langle A_o{}^a(k)\; A_o{}^k(p)\rangle = \frac{i}{(2\pi)^4}\,\delta_{ab}\delta^4(p+k)g^2F(\vec{k}) \quad . \tag{4.4}$$

To calculate the Wilson loop, one needs the corresponding propagators in configuration space for imaginary time. Equation (2.15a) with $k_o \to ik_4$ becomes

$$<A_i^{\,a}(x)\ A_j^{\,b}(y)> = \frac{\delta_{ab}}{(2\pi)^4}\int d^4k\ e^{-ik\cdot(x-y)}\ P_{ij}(\vec{k})\ \frac{(1+F_1(\vec{k}))}{k_4^{\,2}+\omega(\vec{k})^2}$$

$$= \frac{\delta_{ab}}{2(2\pi)^3}\int d^3k\ P_{ij}(\vec{k})A(\vec{k})\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}e^{-\omega(\vec{k})|x_4-y_4|}\quad, \tag{4.5}$$

and (4.4) with $A_o = iA_4$ is

$$<A_4^{\,a}(x)\ A_4^{\,b}(y)> = \frac{g^2}{(2\pi)^4}\ \delta_{ab}\int d^4k\ e^{-ik\cdot(x-y)}F(\vec{k})$$

$$= \frac{g^2}{(2\pi)^3}\ \delta(x_4-y_4)\int d^3k\ e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}F(\vec{k}) = \delta x_4-y_4)V(\vec{x}-\vec{y})\ . \tag{4.6}$$

The gluon energy is $\omega(\vec{k}) = \alpha\lambda + [(1+\bar{F}_1)(k^2+\bar{F}_2)]^{1/2}$. Since the propagator for transverse gluons is proportional to $\exp(-\alpha\lambda|x_4-y_4|)$, physical gluons do not propagate over finite time intervals. When $x_4 = y_4$, the propagator is finite in the $\lambda \to \infty$ limit.

Since transverse gluons do not propagate over finite time intervals, the spacelike parts of the loop at $\tau = T$ and at $\tau = 0$ are disconnected from each other. The loop factors into

$$L = G(R)^2<tr\{P\ \exp(i\int_o^T d\tau\, A_4(\vec{n}R\tau))P\ \exp(i\int_T^o A_4(0,\tau)d\tau)\}> \tag{4.7}$$

where

$$G(R) = \frac{1}{N}<tr\{\exp(i\int_o^R d\sigma\ A_n(\vec{n}\sigma,0))\}> \tag{4.8}$$

$G(R)$ is τ independent. Although a complete calculation of $G(R)$ is difficult, one does find that term-by-term $G(R) \to 1$ as $R \to \infty$. Confinement comes from the remaining VEV in (4.7). There are contributions from equal τ propagators acting across the loop from $\vec{x} = \vec{n}R$ to $\vec{x} = 0$. In addition it is possible for a τ-like gluon to be emitted and re-absorbed by the same side, if the contracted fields are nearest neighbors. A typical term in the expansion of the VEV in (4.8) is

$$L_{nm} = (i)^{n+m} \operatorname{tr}[\lambda^{a_1} \ldots \lambda^{a_n} \lambda^{b_1} \ldots \lambda^{b_m}]$$

$$\left(\int_o^T dy_1 \int_o^{y_1} dy_2 \ldots \int_o^{y_{n-1}} dy_n\right)\left(\int_T^o dz_1 \int_T^{z_1} dz_1 \ldots \int_T^{z_{m-1}} dz_m\right) \tag{4.9}$$

$$\langle A_4^{a_1}(\vec{n}R,y_1) \ldots A_4^{a_n}(\vec{n}R,y_n)\; A_4^{b_1}(0,z_1) \ldots A_4^{b_4}(0,z_m)\rangle .$$

A nearest neighbor contraction has the form

$$\int_o^{y_{i-1}} dy_i \int_o^{y_i} dy_{i+1} \int_o^{y_{i+1}} dy_{i+2}\; \lambda^{a_i}_{is}\, \lambda^{a_{i+1}}_{sj} \langle A_4^{a_i}(\vec{n}R,y_i) A_4^{a_{i+1}}(\vec{n}R,y_{i+1})\rangle$$

$$= \frac{2(N^2-1)}{N}\, \delta_{ij}\, \frac{1}{2}\, V(0) \int_o^{y_{i-1}} dy_i \int_o^{y_i} dy_{i+2} \quad . \tag{4.10}$$

The λ^a matrices satisfy $(\lambda^a\lambda^a)_{ij} = 2(N^2-1)/N\ \delta_{ij}$. The chain of m ordered integrals is shortened by one for each adjacent pair contraction. The integral that marks the position of the contraction is empty in the sense that its argument is unity. Contractions along a side of the loop that are not between adjacent fields vanish due to the conflict between time ordering and the instantaneous propagator.

The second multiple integral in (4.9) is re-ordered to match the form of the first.

$$\int_T^o dz_1 \int_T^{z_1} dz_2 \ldots \int_T^{z_{m-1}} dz_m = (-1)^m \int_o^T dz_m \int_o^{z_m} dz_{m-1} \ldots \int_o^{z_2} dz_1 \quad . \tag{4.11}$$

Contractions across the loop from $\vec{x} = nR$ to $\vec{x} = 0$ connect equal times. Since the two sides are time ordered, contractions cannot cross each other. The equal time propagators are like rungs on a ladder. A generic term in L_{nm} has r rungs, s nearest neighbor contractions on one side, and t nearest neighbor contractions on the other side.

$$L_{nm} = (-1)^m (i)^{n+m} \sum_{s,t} \left[\frac{\rho V(0)}{2}\right]^{s+t} N[\rho V(R)]^r\, I(r,s,t) \quad , \tag{4.12}$$

where $n = 2s + r$ and $m = 2t + r$. The gauge group factor is $\rho = 2(N^2-1)/N$. The residual integral $I(r,s,t)$ includes a sum over all possible orderings of the r rungs, s contractions on one side, and t contractions on the other. Each contraction across the loop from $\vec{x} = nR$ to $\vec{x} = 0$ produces a factor of V(R). The trace of the product of color matrices collapses to $N\rho^{s+t+r}$. Re-ordering of the integral in (4.11) means that rung contractions are between adjacent pairs of λ matrices in the product in (4.9). For example, if s=t=0, equal time rung contractions set $a_n = b_1$, $a_{n=1} = b_2$, etc. When $s \neq 0$ and $t \neq 0$, pairs of adjacent λ matrices are first eliminated by (4.10).

Computation of I(r,s,t) begins with a simple example. If r = 1, s = 1, then the integral on one side has the form

$$\int_o^T dy_1 \int_o^{y_1} dy_2 \int_o^{y_2} dy_3 [F(y_1) + F(y_2) + F(y_3)] = \frac{T^2}{2} \int_o^T F(x)\,dx \quad . \tag{4.13}$$

In general summing over all possible ordering of s adjacent contractions along a side of length n = r + 2s replaces the original ordered integral by one of length r multiplied by $T^s/s!$. Thus, I(r,s,t) is reduced to

$$I(r,s,t) = \frac{T^s}{s!}\frac{T^t}{t!} I(r,0,0) \quad . \tag{4.14}$$

The remaining integral I(r,0,0) is

$$I(r,0,0) = (\int_o^T dy_1 \ldots \int_o^{y_{r-1}})(\int_o^T dw_1 \ldots \int_o^{w_{r-1}} dw_r)\delta(y_1-w_1) \ldots \delta(y_r-w_r) = \frac{T^r}{r!} \quad . \tag{4.15}$$

When the various factors are brought together and summed over n and m, L becomes

$$L = NG(R)^2 \sum_{s,t,r=0}^{\infty} \frac{1}{s!} [-\frac{\rho TV(0)}{2}]^s \frac{1}{t!} [-\frac{\rho TV(0)}{2}]^t \frac{1}{r!} [\rho TV(R)]^r = NG(R)^2 e^{-\rho[V(0)-V(R)]T} \quad . \tag{4.16}$$

If $F(\vec{k})$ is given by (3.6) with n = 2,

$$\rho V(R) = -\frac{g^2(N^2-1)}{4\pi N} fr \quad , \tag{4.17}$$

and the exponential in the Wilson loop has the area dependence that signals confinement.

The simplicity and success of this calculation depends on the instantaneous propagation of the modified Coulomb interaction. Finite propagation time would lead to crossed rungs and longer range (in τ) interactions along a side of the loop. If physical gluons propagate over finite time intervals there are additional complications. The general discussion of the λ dependence of amplitudes indicates that there are no contributions with internal three and four gluon vertices or with internal quark loops.

V. Bound States

There is a large gap between a general proof of confinement and the derivation of explicit bound state equations. The mean field theory model incorporates the full machinery of field theory. Thus, it is possible to derive Bethe-Salpeter equations for both gluon-gluon and quark-quark states. Although the three quark calculation is possible,[20] it is not discussed here.

The need to consider three different gluon propagators (Eq. (2.15)) is a complication in the analysis of gluon-gluon scattering. A new index which labels field type is added to the gluon field: $A_{i1}{}^{a}(\vec{p}) = A_{i}{}^{a}(\vec{p})$ and $A_{i2}{}^{a}(\vec{p}) = P_{i}{}^{a}(\vec{p})$. The gluon propagator is two-by-two matrix in type space. The Wilson loop calculation identifies the modified Coulomb interaction as the source of confinement. With the new notation the gluon Coulomb Hamiltonian is

$$H_C = \frac{g^2}{8} (2\pi)^4 \int d^4 1 \; d^4 2 \; d^4 3 \; d^4 4 \; \delta^4(1 + 2 + 3 + 4)$$
$$[f_{abc} \vec{A}_r{}^{b}(1) \cdot \vec{A}_s{}^{c}(2)] M_{rs} \; F(-\vec{1}-\vec{2}) \tag{5.1}$$
$$[f_{ade} \; \vec{A}_t{}^{d}(3) \cdot \vec{A}_u{}^{e}(4) M_{tu}] \quad ,$$

where

$$M_{rs} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.2}$$

A gluon line in a Feynman diagram carries momentum and three discrete indices.

The n-rung, color singlet ladder amplitude with Coulomb exchange for the process $[W/2+p,i,b,r] + [W/2-p,k,d,t] \to [W'/2-k,j,c,s] + [W'/2+k,m,e,u]$ is

$$M^{(n)}(p,W,k,W') = -(2\pi)^8\delta^4(W+W') \quad \frac{\delta_{bd}\delta_{ce}}{N^2-1}\,\Psi^{(n)}(p,k;W) \quad , \tag{5.3}$$

where

$$\Psi^{(n+1)}{}^{ir}_{kt}(p,k;W)^{js}_{mu} = \frac{ig^2N}{(2\pi)^4}\int d^4x\; F(\vec{p}-\vec{x})\; P_{ii'}(\tfrac{\vec{W}}{2}+\vec{x})\; P_{kk'}(\tfrac{\vec{W}}{2}-\vec{x})$$
$$M_{rr''}D_{r''r'}(\tfrac{W}{2}+x)\; M_{tt''}D_{t''t'}(\tfrac{W}{2}-x)\Psi^{(n)}{}^{i'r'}_{k't'}(x,k;W)^{js}_{mu} \quad , \tag{5.4}$$

and

$$\psi^{(1)}{}^{ir}_{kj}(p,k;W)^{js}_{mu} = \frac{ig^2N}{(2\pi)^4}\;F(\vec{p}-\vec{k})\;M_{rs}M_{tu}\;\delta_{ij}\,\delta_{km} \quad . \tag{5.5}$$

The propagator matrix in type space is

$$D(p) = \frac{1}{d(p)}\begin{vmatrix} 1+F_1(\vec{p}) & -ip_o \\ ip_o & p^2+F_2(\vec{p}) \end{vmatrix} \quad . \tag{5.6}$$

The ladder amplitude $M^{(n)}$ is of order λ in the infrared singular limit. Equations (5.4) and (5.5) can be summed to produce an inhomogeneous Bethe-Salpeter equation for Ψ,

$$\Psi = \sum_{n=1}^{\infty}\Psi^{(n)} = \Psi^{(1)} + K\sum_{n=1}^{\infty}\psi^{(n)} = \Psi^{(1)} + K\Psi. \tag{5.7}$$

Since the Coulomb interaction is a function of three-momentum transfer only, the wave function Ψ is a function of the four-vector momentum W_μ but not of the time components of p_μ or k_μ. It is possible to evaluate the x_o integration in (5.4).

$$\int dx_o\,\frac{D_{r''r'}(\frac{W}{2}+x)\;D_{t''t'}(\frac{W}{2}-x)}{d(\frac{W}{2}+x)\;d(\frac{W}{2}-x)} = \frac{\pi i}{2}\left[\frac{\tilde{D}_{r''r'}(\frac{\vec{W}}{2}+\vec{x})\;\tilde{D}_{t''t'}(\frac{\vec{W}}{2}-\vec{x})}{\Sigma(\vec{x},\vec{W})+W_o}\right.$$
$$\left.+\frac{\tilde{D}^*_{r''r'}(\vec{W}/2+\vec{x})\;\tilde{D}^*_{t''t'}(\vec{W}/2-\vec{x})}{\Sigma(\vec{x},\vec{W})-W_o}\right] \quad , \tag{5.8}$$

where

$$\tilde{D}(\vec{p}) = \begin{pmatrix} A(\vec{p}) & -i \\ i & \frac{1}{A(\vec{p})} \end{pmatrix} , \qquad (5.9)$$

and $\Sigma(\vec{x},\vec{W}) = \omega(\vec{W}/2+\vec{x}) + \omega(\vec{W}/2-\vec{x})$, $\omega(\vec{k}) = [(1+F_1(\vec{k}))(k^2+F_2(\vec{k}))]^{1/2}$. The matrix $\tilde{D}(\vec{p})$ is infrared finite, but $\Sigma \simeq 2\alpha\lambda$. The infrared divergence in the interaction $F(\vec{k})$ is cancelled by the divergence in Σ.

Gluon-gluon scattering is not a physical process, except when it occurs as an insertion in a more complicated diagram. On the other hand, bound states are solutions of the homogeneous Bethe-Salpeter equation.

$$\Psi^{ir}_{kt}(\vec{p},\vec{W},W_o) = -\frac{\alpha}{2}\int d^3x\; F(\vec{p}-\vec{x})\; P_{ii'}(\vec{W}/2+\vec{x})\; P_{kk'}(\vec{W}/2-\vec{x})$$

$$\left\{\frac{[M\tilde{D}(\vec{W}/2+\vec{x})]_{rr'}[M\tilde{D}(\vec{W}/2-\vec{x})]_{tt'}}{\Sigma(\vec{x},\vec{W})+W_o} \right. \qquad (5.10)$$

$$\left. + \frac{[M\tilde{D}^*(\vec{W}/2+\vec{x})]_{rr'}[M\tilde{D}^*(\vec{W}/2-\vec{x})]_{tt'}}{\Sigma(\vec{x},\vec{W})-W_o}\right\} \Psi^{i'r'}_{k't'}(\vec{x},\vec{W};W_o) \quad .$$

This equation must have a finite, non-trivial, infrared limit if color bound states are to exist. There are awkward kinematic factors in (5.10) which reflect the different dimensions of the fields $A_i^{\,a}(p)$ and $P_i^{\,a}(p)$. It is helpful to make the change

$$\Psi^r_t(\vec{p},\vec{W};W_o) = \theta_{rr'}(\vec{W}/2+\vec{p})\;\theta_{tt'}(\vec{W}/2-\vec{p})\;\Phi^{r'}_{t'}(\vec{p},\vec{W};W_o) \quad , \qquad (5.11)$$

where

$$\theta(\vec{p}) = \begin{pmatrix} 1 & 0 \\ 0 & A(\vec{p}) \end{pmatrix} \quad . \qquad (5.12)$$

The propagator $\tilde{D}(\vec{p})$ simplifies.

$$\tilde{D}'(\vec{p}) = \Theta(\vec{p})\;\tilde{D}(\vec{p})\;\theta(\vec{p}) = 2A(\vec{p})\;\frac{1}{2}\;[1+\sigma_2] \quad , \qquad (5.13)$$

where σ_2 is a Pauli spin matrix. Since $P_+ = [1+\sigma_2]/2$ is a projection matrix, $\tilde{D}'(\vec{p})$ projects out a certain amplitude in type space. The complex conjugate matrix $D'(p)^*$ is proportional to $P_- = [1-\sigma_2]/2$. If

$$\tilde{M}(\vec{p},\vec{x}) = O^{-1}(\vec{p})\, M O^{-1}(\vec{x}) = \begin{pmatrix} 0 & -\dfrac{1}{A(\vec{x})} \\ \dfrac{1}{A(\vec{p})} & 0 \end{pmatrix} , \tag{5.14}$$

the homogeneous Bethe-Salpeter equation becomes

$$\begin{aligned} \Phi^{ir}_{kt}(\vec{p},\vec{W};W_o) = -2\alpha \int d^3x\, F(\vec{p}-\vec{x})\, P_{ii'}(\vec{W}/2 + \vec{x})\, P_{kk'}(\vec{W}/2 - \vec{x}) \\ A(\vec{W}/2 + \vec{x})\, A(\vec{W}/2 - \vec{x})\, \tilde{M}(\vec{W}/2 + \vec{p}, \vec{W}/2 + \vec{x})_{rr'}\, \tilde{M}(\vec{W}/2 - \vec{p}, \vec{W}/2 - \vec{x})_{tt'} \\ \left\{\frac{(P_+)_{r'r''}(P_+)_{t't''}}{\Sigma(\vec{x},\vec{W}) + W_o} + \frac{(P_-)_{r'r''}(P_-)_{t't''}}{\Sigma(\vec{x},\vec{W}) - W_o}\right\} \Phi^{i'r''}_{k't''}(\vec{x},\vec{W};W_o) \end{aligned} \tag{5.15}$$

The plethora of indices makes (5.15) appear complicated.

In the infrared limit $\Sigma = 2\alpha\lambda + \bar{\Sigma} \simeq 2\alpha\lambda$ and $F(\vec{p}-\vec{k}) = \lambda\delta^3(\vec{p}-\vec{k})$. Moreover, $A(\vec{p})\tilde{M}(\vec{p},\vec{p}) = -i\sigma_2$. Since $\sigma_2 P_\pm = \pm P_\pm$, the condition for infrared finite solutions is

$$\begin{aligned} \Phi^{ir}_{kt}(\vec{p},\vec{W};W_o) = P_{ii'}(\vec{W}/2 + \vec{p})P_{kk'}(\vec{W}/2 - \vec{p}) \\ \{(P_+)_{rr'}(P_+)_{tt'} + (P_-)_{rr'}(P_-)_{tt'}\}\, \Phi^{i'r'}_{k't'}(\vec{p},\vec{W};W_o) \quad . \end{aligned} \tag{5.16}$$

The wave function is transverse in spin indices. In type space two of four possible amplitudes must vanish. Cancellation of the infrared singularity with $\Phi \neq 0$ is possible only if the coupling constant α in (5.15) is identical to the constant α in the divergent part of Σ. The color singlet channel is the only one in which the equality holds. Using (5.16), one can write Φ in the form

$$\begin{aligned} \Phi^r_t = (\Sigma + W_o)\Theta_+ \chi^{(+)}_r \chi^{(+)}_t \\ + (\Sigma - W_o)\Theta_- \chi^{(-)}_r \chi^{(-)}_t \quad , \end{aligned} \tag{5.17}$$

with

$$\chi^{(\pm)} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{5.18}$$

The ++ and -- components of (5.15) can be projected out to produce a pair of infrared finite integral equations for two gluon bound states.

$$[\bar{\Sigma}(\vec{p},\vec{W}) + W_o]\Theta_+(\vec{p},\vec{W};W_o) = \alpha \int d^3x\ \bar{F}(\vec{p}-\vec{x})$$
$$[U^{(+)}(\vec{p},\vec{x};\vec{W})\Theta_+(\vec{x},\vec{W};W_o) + U^{(-)}(\vec{p},\vec{x};\vec{W})\Theta_-(\vec{x},\vec{W};W_o)] \quad (5.19a)$$

$$[\bar{\Sigma}(\vec{p},\vec{W}) - W_o]\Theta_-(\vec{p},\vec{W};W_o) = \alpha \int d^3x\ \bar{F}(\vec{p}-\vec{x})$$
$$[U^{(-)}(\vec{p},\vec{x};W)\Theta_+(\vec{x},\vec{w};W_o) + U^{(+)}(\vec{p},\vec{x};W)\Theta_-(\vec{x},\vec{W};W_o)], \quad (5.19b)$$

$\bar{\Sigma}(\vec{p},\vec{W})$ is the infrared finite part of $\Sigma(\vec{p},\vec{W})$. Spin indices have been suppressed and

$$U^{(\pm)}(\vec{p},\vec{x};\vec{W}) = \frac{1}{2}\ \frac{[A(\vec{W}/2 + \vec{x}) \pm A(\vec{W}/2 + \vec{p}][A(\vec{W}/2 - \vec{x}) \pm A(\vec{W}/2 - \vec{p})]}{A(\vec{W}/2 + \vec{p})\ A(\vec{W}/2 - \vec{p})} \quad (5.20)$$

If each $A(\vec{p})$ is replaced by $1/p$, equation (5.19) is identical to the Tamm-Dancoff equation that appeared in an earlier paper.[3]

The Bethe-Salpeter equation, unlike the Tamm-Dancoff equation, can be renormalized.[21] Using the set of renormalization constants of Appendix D, one finds $\bar{\Sigma} = [(1+F_{1R}(\nu))(1+(F_{2R}(\nu) - F_{2R}(0))/\nu^2)]^{1/2}\ \bar{\Sigma}_R = Z_\Sigma \bar{\Sigma}_R$. If the energy W_o is renormalized by the same factor, all dependence on renormalization constants disappears from (5.19).

The Tamm-Dancoff version of (5.19) was solved in reference 3. A relativistic WKB method was used for high lying energy levels. The ground state and first few excited states were determined numerically. The WKB calculation has been repeated for the Bethe-Salpeter equation kinematics and the results are unchanged. The conclusion is that if $F(\vec{p}-\vec{x})$ confines according to the Wilson criterion, then (5.19) has an infinite number of bound state solutions. In the limit of large principle quantum number N', the bound state energy is proportional to $N'^{1/2}$, if $F(\vec{k}) \propto k^{-4}$ as $k \to 0$.

The derivation of the Bethe-Salpeter equation for quarks follows the pattern laid down for gluons. One considers ladder diagrams for quark-antiquark scattering. The color singlet channel is projected out. Summation of the ladder diagrams leads to an inhomogeneous integral equation. The homogeneous equation for bound states is

$$\Psi_{\alpha\beta}(\vec{p},\vec{W};W_o) = \frac{i\alpha'}{(2\pi)} \int d^4x \; F(\vec{p}-\vec{x})$$
$$[\gamma_o S(-W/2-x)]_{\alpha\alpha'} \; \Psi_{\alpha'\beta'}(\vec{x},\vec{W};W_o)\,[S(W/2-x)\gamma_o]_{\beta\beta'} \quad . \tag{5.21}$$

Again it is possible to do the x_o integration. The quark counterparts of the gluon projection operators are the positive and negative energy projection operators

$$\Lambda_{\pm}(\vec{k}) = \frac{1}{2}\,[1 \pm \frac{\gamma_o\vec{\gamma}\cdot\vec{k}(1+G_1(k)) + \gamma_o m(1+G_2(k))}{2E(k)}] \quad . \tag{5.22}$$

The quark Bethe-Salpeter equation is

$$\Psi(\vec{p},\vec{W};W_o) = \alpha' \int d^3x \; F(\vec{p}-\vec{x})$$
$$\{\frac{\Lambda_-(\vec{W}/2-\vec{x})\;\Psi(\vec{x},\vec{W};W_o)\;\Lambda_+(\vec{W}/2-\vec{x})}{\Sigma_f + W_o} \tag{5.23}$$
$$+ \frac{\Lambda_+(-\vec{W}/2-\vec{x})\;\Psi(\vec{x},\vec{W};W_o)\;\Lambda_-(\vec{W}/2-\vec{x})}{\Sigma_F - W_o}\} \quad ,$$

and $\Sigma_f = E(\vec{W}/2+\vec{x}) + E(\vec{W}/2-\vec{x}) = \alpha'\lambda + \bar{\Sigma}_f$. Using (3.17) and (3.18) one finds that the projection operators are infrared finite.

A finite equation is possible only if there is a cancellation between Σ_f and $F(\vec{p}-\vec{x}) \propto \lambda\delta^3(\vec{p}-\vec{x})$. In all qq and $\bar{q}$q channels, except the color singlet one, infrared cancellation leads to $\Psi \equiv 0$. The color singlet wave function becomes $\Psi = \Psi_{+-} + \Psi_{-+}$, where the subscript indicates projection with Λ_+ or Λ_-. If

$$\Psi_{\substack{+-\\-+}} = [\Sigma_f \pm W_o]\Phi_{\substack{+-\\-+}} \quad , \tag{5.24}$$

the bound state equations are

$$[\bar{\Sigma}_f(\vec{p},\vec{W}) - W_o]\Phi_{-+}(\vec{p},\vec{W},W_o) = \Lambda_-(-\vec{W}/2-\vec{p})\{\alpha' \int d^3x \; \bar{F}(\vec{p}-\vec{x})$$
$$[\Phi_{-+}(\vec{x},\vec{W};W_o) + \Phi_{+-}(\vec{x},\vec{W};W_o)]\}\Lambda_+(\vec{W}/2-\vec{p}) \quad , \tag{5.25a}$$

$$[\bar{\Sigma}_f(\vec{p},\vec{W}) + W_o]\Phi_{+-}(\vec{p},\vec{W};W_o) = \Lambda_+(-\vec{W}/2-\vec{p})\{\alpha' \int d^3x \; \bar{F}(\vec{p}-\vec{x})$$
$$[\Phi_{-+}(\vec{x},\vec{W};W_o) + \Phi_{+-}(\vec{x},\vec{W};W_o)]\}\Lambda_-(\vec{W}/2-\vec{p}) . \tag{5.25b}$$

In the center of momentum frame where $\vec{W} = 0$, these equations are identical to the Tamm-Dancoff equations of reference 3. There it was shown, both numerically and with the WKB approximation, that there exists an infinite number of bound states. One expects such a spectrum in a confining theory. Again it is possible to renormalize the bound state equations.

The mean field model produces finite energy bound states from infinite energy constituents. Moreover, the three-momentum of the bound state need not be zero. There are no in principle restrictions to heavy quarks. The relationship of the renormalized quark mass to the conventional quark mass is not clear. A detailed discussion of the solutions of (5.19) and (5.25) is beyond the scope of this paper. Corrections to the wave equations cannot, according to Appendix C, involve three gluon, four gluon, or gluon-quark-quark vertices. Gluon exchange between the Coulomb rungs of the ladder is possible, if one includes the multi-gluon terms in the operator product expansion of $F_{ab}(\vec{p},\vec{k};x)$. In addition, non-planar kernels are possible, although suppressed in the $N \to \infty$ (of SU(N)) limit.

Finite energy bound states constitute a new class of particle that must be added to the Feynman rules of the mean field theory model. One should consider diagrams where bound states occur in intermediate states. The $\lambda \to \infty$ limit does not appear to suppress such contributions. The full apparatus of the field theory allows normalization of the Bethe-Salpeter wave functions, a requirement for the derivation of generalized Feynman rules. Some results are given in Appendix G.

VI. Anomalous Interactions

The Coulomb gauge Hamiltonian given in (2.1) has two terms which do not appear if QCD is quantized naively. The so-called anomalous interactions appear when the theory is first quantized in a non-singular gauge and then transformed to the Coulomb gauge.[7] The need for $V_1(A)$ was recognized by Schwinger[18] many years ago. The generators of Lorentz boosts do not satisfy the Poincare algebra unless $V_1(A)$ exists. Christ and Lee[7] started with the temporal gauge, quantized the theory, and then transformed to the Coulomb gauge. Although the distinction is ambiguous, the $V_1(A)$ term arises when explicit dependence on the Faddeev-Popov determinant is eliminated with commutation relations. The second anomalous term, $V_2(A)$, appears when quantum operators are Weyl ordered. The Weyl ordering is necessary for deriving the correct Hamiltonian to use in the path integral. Recent calculations demonstrate that these extra terms are necessary for renormalizability.[15] They cancel divergences that arise in higher order (g^4 and beyond). Only with these extra interactions is one able to reproduce the results of standard calculations. On the other hand, these interactions have been ignored in the development of the mean field model. The mean field model is designed to describe the low momentum limit of QCD. Thus, it is possible that the extra terms in the Hamiltonian are irrevelant in the infrared limit. This section is devoted to a justification of the neglect of anomalous interactions.

An important feature of the mean field model is the balance between the infrared divergence in quark and gluon self energies and the divergence in the modified Coulomb interaction. If the anomalous interactions also produce infrared divergences, then there could be cancellations with the standard terms and confinement would be a spurious effect. This is unlikely. There is no direct coupling to quarks in either $V_1(A)$ or $V_2(A)$. (See Appendix A) Quark amplitudes are insensitive to these terms. Gluonic corrections to quark amplitudes necessarily involve gluon propagators which vanish in the infrared limit. Since the canonical momentum field $P_i^a(p)$ does not couple directly, the same logic suggests that bound states in the gluon sector are unaffected. The VEV of the modified Coulomb interaction remains the only candidate for the source of the confining interaction. Moreover, this VEV is insensitive to anomalous interactions for the same reason that quark amplitudes are insensitive.

The only way in which $V_1(A)$ and $V_2(A)$ can scuttle the mean field model is for them to generate effective two- and/or four-gluon interactions which are infrared singular. The question is difficult to settle with absolute finality because of the complexity of reliable, non-perturbative calculations even within the restrictions of the mean field model. Since the origin of $V_1(A)$ seems more fundamental, I analyze it first. An effective two-gluon coupling can be derived from (A2). The structure of $V_1(A)$ is that of a loop composed of D-type Coulomb propagators. As shown in Figure 4a, the loop is bisected by a fictitious gluon. A fictitious gluon carries the momentum, color, and spin of a physical gluon, but does not introduce a momentum dependent propagator. Using the operator product expansion for each $D_{ab}(\vec{p},\vec{k};x)$ in (A2), one finds the set of diagrams in Figure 4b. Since the operator product expansion does not allow gluon lines to re-couple to the same Coulomb line, gluons are either external or couple to the opposing Coulomb line. Non-planar diagrams do not survive the $N\to\infty$ limit. Moreover, only planar diagrams will cancel the planar contributions of the standard Coulomb Hamiltonian. The diagrams of Figure 4b can be summed to give Figure 4c. In the limit that all vertices are replaced by points, the two gluon effective action arising from $V_1(A)$ is

$$\int dt V_1(A) = -\frac{g^4N^2}{16(2\pi)^2}\int d^4p\, d^4k\,\delta^4(p+k)\; A_i{}^a(p)\; A_j{}^a(k)$$
$$\{\int d^3s\, s_i(p-s)_m\; D(\vec{s})\; D(\vec{p}-\vec{s})\; \int d^3t\, t_j(k-t)_m\; D(\vec{t})\, D(\vec{k}-\vec{t})\} \quad . \qquad (6.1)$$

This result is to be compared with the $A_i{}^a(p)A_i{}^a(k)$ term in (2.9). The s and t integrals are infrared convergent and ultraviolet divergent. They presumably cancel a g^4 divergence arising from the standard Coulomb Hamiltonian. If the s integral is cut off at infinity and evaluated with $D(\vec{s}) \propto m^{1/2}s^{-5/2}$ $(A(0) = 1/m)$, one finds it is proportional to $\delta_{im}p^o$. The t integral is identical, and the net result is a constant.

$$\int dt V_1(A) \approx g^4N^2 \int d^4p\, d^4k\,\delta^4(p+k)\; m^2A_i{}^a(p)\; A_i{}^a(k) \quad . \qquad (6.2)$$

The constant may cancel in the renormalization process. Unlike the $F_2(\vec{k})$ term in (2.9), the effective action in (6.1) is infrared finite. There is no divergence proportional to λ. For comparison the finite part of $F_2(\vec{k})$, after renormalization, is proportional to k^{-1}, if the Coulomb interaction $F(\vec{k}) \propto k^{-4}$.

Although (6.1) does not have an infrared singularity, there is the possibility that a singularity could develop from non-perturbative effects at the fictitious gluon vertices. The discussion in Appendix E indicates that corrections from a finite number of gluons spanning a vertex cannot alter the infrared behavior. The analysis of Appendix F can be used to calculate non-perturbative corrections corresponding to Figure 4c. An approximation to (A2) which includes (6.1) is

$$\begin{aligned} \int dt V_1(A) &= -\frac{g^4}{8(2\pi)^2} \int d^3p_1\, d^3p_2 d^2k_1\, d^3k_2 dx_1\, dx_2 dy_1\, dy_2 d^4s_1\, d^4s_2 d^4t_1\, d^4t_2 \\ &\int d^4s d^4t\, A^{e'}(s)_i A^{d'}(t)_j\, \delta^3(\vec{p}_1+\vec{p}_2+\vec{k}_1+\vec{k}_2)\, \delta^3(\vec{s}+\vec{s}_1+\vec{s}_2)\, \delta^3(\vec{t}+\vec{t}_1+\vec{t}_2) \\ &\quad \delta(s_o+t_o+x_1+x_2+y_1+y_2)\, f_{ade} f_{cbe} f_{xe'y} f_{w'd'z}\, (2\pi)^{12}\, \vec{k}_1\cdot\vec{k}_2 (s_1)_i (s_2)_j \\ &\langle D_{yb}(\vec{s}_2,\vec{k}_1;x_2) D_{cw}(\vec{p}_2,\vec{t};y_1)\rangle\langle D_{ax}(\vec{p}_1,\vec{s}_1;x_1) D_{zd}(\vec{t}_2,\vec{k}_2;y_2)\rangle \end{aligned} \tag{6.3}$$

The VEV of a pair of D's is evaluated in Appendix F. In that limit (6.3) is

$$\begin{aligned} \int dt V_1(A) &= -\frac{4g^4}{(2\pi)^2} \frac{N^2}{[N^2-1]^2} \int d^4p\, d^4k A_i{}^a(p)\, A_j{}^a(k)\delta^4(p+k) \\ &\{[\int d^3s_1 d^3s_2\, d^3W [W+s_2]_i [W-s_2]_m\, \Psi(\vec{s}_1,\vec{s}_2;\vec{W}) \\ &\quad D(\vec{W}+\vec{s}_2)\, D(\vec{W}-\vec{s}_2)\, D(\vec{W}+\vec{s}_1) D(\vec{W}-\vec{s}_1)] \\ &[\int d^3t_1 d^3t_2\, d^3W' [W'-t_2]_j [W'+t_2]_m\, \Psi(\vec{t}_1,\vec{t}_2;\vec{W}') \\ &\quad D(\vec{W}'+\vec{t}_1)\, D(\vec{W}'-\vec{t}_1)\, D(\vec{W}'+\vec{t}_2)\, D(\vec{W}'-\vec{t}_2)] \\ &\quad \delta^3(s_1+s_2-t_1-t_2)\, \delta^3(p+t_1+W'-s_2-W)\} \quad . \end{aligned} \tag{6.4}$$

The wave function $\Psi(\vec{p},\vec{k};\vec{W})$ is defined in Appendix F. It is the color singlet wave function for two D-type Coulomb lines bound together by gluon exchange. Without solving for Ψ, I cannot conclude that there are no infrared singularities buried in (6.4). The singular function $F(\vec{k})$ is a particular integral over $\Psi(\vec{p},\vec{k};0)$. If $\Psi(\vec{p},\vec{k};\vec{W})$ is a homogeneous function with the dimensions necessary to make $F(\vec{k})$ singular, then a simple scaling of all momenta in (6.4) by p suggests that the factor in curly brackets is proportional to p^2. There is no evidence of singular behavior in the $p \to 0$ limit.

The color factor $N^2/[N^2-1]^2$ vanishes in the $N \to \infty$ limit. Color singlet wave functions do not couple strongly across a vertex. Other color configurations might not be suppressed, but there is no evidence for singularities in those channels. The two Green's functions in the definition of $F_{ab}(\vec{p},\vec{k};x)$ couple in a way that enhances the infrared behavior. The two Green's functions in $V_1(A)$ couple differently.

It is possible to analyze the three terms of $\int V_2(A)$ in the same way. Again, there are fictitious and real gluon lines. The obvious infrared singularity from $\langle F_{ab}(\vec{p},\vec{k};x)\rangle = \lambda\delta_{ab}\delta(\vec{p}+\vec{k})\delta(x)F(\vec{p})$ is suppressed. Either it leads to an irrelevant c-number, or the projection operators for fictitious gluons force the terms to vanish when $F(\vec{p})$ is replaced by $\lambda\ \delta(\vec{p})$. Wherever it appears, $F(\vec{p})$ can be replaced by $\bar{F}(\vec{p})$. The contribution of $\int V_2(A)$ to the two gluon action, Figure 5, in the point vertex limit is

$$
\begin{aligned}
\int V_2(A)\,dt = &-\frac{g^2N^2}{8(2\pi)^2}\int d^4p\ d^4k\,\delta^4(p+k)A_i{}^a(p)\ A_j{}^a(k) \\
&\{\int d^3s\ d^3t\{s_i s_j\ \mathrm{tr}[P(\vec{s})\ P(\vec{s}-\vec{t})]\ \frac{d}{dg}\ [g^3\ D(\vec{s})\ D(\vec{s})\ D(\vec{s}+\vec{p})] \\
&\quad + s_j[P(\vec{p}+\vec{t})\cdot P(\vec{s}+\vec{p}+\vec{t})\cdot\vec{t}]_i\ gD(\vec{t})\ \frac{d}{dg}\ [g^2D(\vec{s})\ D(\vec{s}+\vec{p})] \\
&+ 2s_j[P(\vec{p}+\vec{s})\cdot P(\vec{p}+\vec{s}+\vec{t})\cdot\vec{t}]_i\ g^2D(\vec{s})\ D(\vec{s}+\vec{p})\ F(\vec{t}) \\
&+ \frac{1}{2}\ [P(\vec{p}+\vec{s})\cdot\vec{t}]_i\ [P(\vec{t}-\vec{p})\cdot\vec{s}]_j\ g^2D(\vec{t})\ D(\vec{s})\ F(\vec{p}+\vec{s}-\vec{t})\}\ ,
\end{aligned} \tag{6.5}
$$

where $d[gD(\vec{p})]/dg = F(\vec{p})$. The first s integral in the curly brackets is infrared divergent. However, the t integral produces a multiplicative ultraviolet divergent factor. This term must explicitly cancel a higher order correction arising from the standard Coulomb Hamiltonian. (See Appendix E) One can explicitly check that $F(\vec{p})$ is replaceable by $\bar{F}(\vec{p})$ everywhere else in (6.5). In addition, one can show that in the infrared limit, the other s and t integrals are finite and scale to produce an overall k^{-1} dependence for the factor in curly brackets. Although there is no infrared divergence, the $k \to 0$ behavior matches that of $F_2(\vec{k})$ in (2.9). The g^4 correction to $F_2(\vec{k})$ cannot be renormalized without the g^4 terms from $\int V_2(A)$. It is not surprising that the infrared behavior of the two terms matches. Together they produce a finite higher order correction to the calculation of the gluon propagator function $A(\vec{p})$. Vertex corrections should not alter this conclusion.

Since the contribution to the two gluon effective action is infrared finite, there is no reason to expect that the anomalous interactions generate singular multi-gluon vertices. Wherever it occurs in the operator product expansion of $V_2(A)$, $F(\vec{p})$ can be replaced by $\bar{F}(\vec{p})$. The coupling to gluons suppresses the infrared singularity. Thus, an obvious source of infrared divergence is removed. The only other source of infrared divergence would be the binding of Coulomb lines in a singular, color singlet configuration. The analysis of $V_1(A)$ showed this possibility is not realized at vertices. Multi-gluon interactions derived from $V_2(A)$ may well have a singular momentum dependence in the infrared limit. This dependence matches that of higher order terms from the standard interactions and is necessary for renormalization. The effects of multi-gluon vertices are suppressed by the vanishing of the propagator for physical gluons. Therefore, the only place where the anomalous interactions are relevant is in higher order (g^4 and beyond) corrections to the $\langle AA \rangle$ propagator. The fundamental equations of the mean field model are not sensitive to such corrections, if their momentum dependence matches that of the standard interactions.

VII. Discussion

This paper presents a study of quantum chromodynamics in the $\vec{p} \to 0$, infrared, limit. Conventional perturbation theory is useless in this domain. Progress is possible only if assumptions are made about the physics of the infrared limit. The important experimental observation is that colored states do not exist as independent entities. One possible mechanism to explain this fact is developed in this work. Quarks and gluons are not observed because there is an infrared singular interaction which endows them with infinite energy after renormalization has removed ultra-violet infinities. A candidate for the source of the singularity is identified. One is led naturally to an explicit mechanism by which color singlet bound states of finite energy can be formed from infinite energy constituents. The mean field model is distinguished by the fact that confinement by an infrared singularity is self consistent. When QCD is analyzed subject to the hypothesized infrared behavior, exactly that behavior emerges. Proof of self consistency requires a method for quantizing the theory and a method for carrying out non-perturbative calculations.

The mean field theory starts with path integral quantization in the Coulomb gauge. The non-perturbative nature of the infrared limit is invoked at the beginning. The quantized fields describe non-propagating quarks and gluons. The quadratic, free particle Hamiltonian is modified by counterterms in order to describe the physical fields. The counterterms are calculated by the requirement that the sum of all self energy insertions in particle propagators must vanish. A Hamiltonian based quantization prescription is used to avoid the appearance of the Faddeev-Popov determinant and the need for ghost fields. The trade-off is the necessity for dealing with non-local interactions. Perturbation theory with non-propagating quarks and gluons is simplified by the suppression of large classes of Feynman diagrams. The remaining amplitudes are summed to produce non-perturbative integral equations for the functions of interest. A key element in the non-perturbative process is the use of a variation of the operator product expansion. The infrared singularity originates in the VEV of the operator for the modified Coulomb interaction. When the VEV is actually calculated, it is found to be singular. Quark and gluon energies are found to be infinite. In the limit of large momenta, the infrared finite portions of all functions match approximately the large momentum limits of conventional QCD calculations. The model is subjected to several tests. The Wilson loop calculation provides a general proof of confinement. Bethe-Salpeter equations are derived for color singlet bound states. Correction terms are investigated to establish the stability of the infrared limit. There is no evidence of inconsistency. Thus, as it is presented here, the mean field model appears to provide a framework for the calculation of almost any hadronic amplitude.

Are there problems with this candidate for a model of "everything"? There are serious questions related to the choice of the Coulomb gauge. It is non-covariant and singular. This issue was addressed at length in the introduction. However, it must be emphasized that the infrared limit is when the three-momentum vanishes and the spatial distance becomes large. It is, therefore, not a Lorentz invariant concept. Moreover, bound states composed of a finite number of constituents are not Lorentz covariant. Therefore, a non-covariant gauge is appropriate. The Coulomb gauge is unique in that it identifies and quantizes the physical degrees of freedom of the gauge field. The word physical is used in the sense that were a gluon to appear as a physical state, it would be described by those degrees of freedom. Of course, one would like to transform the Coulomb gauge version of the mean field model to a more general gauge and understand how confinement can occur in those gauges.

A second, but related, question is directed at the fact that confinement is produced by an instantaneous interaction. Again the zero momentum limit can be invoked to suggest that effects due to propagation time should be small. Calculations of relativistic bound states in more conventional situations[22] show that the dominant part of the potential can be treated as instantaneous. Moreover, in the mean field model the nature of the vacuum state is not specified. If the confining potential is due to pressure of the physical vacuum on a bubble of perturbative vacuum, then that potential would, in fact, be instantaneous.

The confining interaction has a second attribute which seems to confound conventional wisdom. It transforms as the fourth component of a four vector rather than as a Lorentz scalar.[23] If quarks with mass are moving slowly in the presence of an instantaneous interaction, there is no essential difference between the fourth component of a four vector and a scalar. An interesting question is which would give the best results for a relativistic system.

There are technical aspects of the mean field model which merit further study. Foremost, of course, is the fact that the singular limit of the interaction function $F(k) \propto k^{-2n}$ is almost, but not quite, consistent with the phenomenologically favored value $n = 2$. The fact that $n=2 \pm i\varepsilon$ is of concern. The model is so tightly constrained that most corrections vanish in the infrared limit. On the other hand, the power dependence of F(k) is fixed in a way that makes it sensitive to correction terms. The infrared limit of both the VEV of the Green's function, D(k), and the propagator function A(k) are determined by matching powers of momenta in the consistency equations. F(k) is subject to the more demanding requirement that the coefficients of the power dependence must match. Higher order corrections do not change the power dependence of D(k) and A(k) but do affect F(k).

Other technical assumptions deserve closer scrutiny. For example, the ordering of infrared and ultraviolet limits is delicate. There are diagrams which are needed to produce the correct behavior in the limit of asymptotic freedom yet which are eliminated by the infrared singularity. Re-ordering of limits would also restore the universality of coupling constant renormalization. In fact, the structure of the mean field model is very thightly constrained by the requirement of self consistency. It is hard to see how any of the hypotheses could be relaxed without destroying the model. Yet one might want less than complete suppression of the quark-quark-gluon vertex. Reordering of limits might help. The final requirement is that no physical quantity can depend on either the infrared cut-off parameter or the details of the ultraviolet regularization procedure.

The whole subject of bound state amplitudes requires further analysis. The Bethe-Salpeter equation should be solved for the bound state spectrum. One can test for the reasonableness of the associated phenomenology. The simple non-relativistic limit[3] suggests the results should not be too different from ad hoc treatments of the bound state problem.[24] Extension to color singlet configurations of three quarks is straightforward.[20] The analysis of Appendix G can be extended to calculate hadronic coupling constants. If weak and electromagnetic interactions are introduced, it should be possible to calculate almost any matrix element. Solutions of the bound state wave equation when the total momentum is not zero should be particularly interesting.

If the mean field model is accepted as a valid approximation to QCD, there are an unlimited number of theoretical applications. Everything is calculable. For example, the quark mass is an inessential parameter. Are there solutions to the counterterm equations for quarks which produce a quark mass when the perturbative mass is zero? If so, one would have a dynamical model for chiral symmetry breaking.[25] One can investigate $\langle F_{\mu\nu}F^{\mu\nu}\rangle$ and $\langle\bar{\Psi}\Psi\rangle$ to study the formation of condensates.[26] In the path integral formalism the nature of the vacuum state is not addressed. Using the effective Hamiltonian for gluons coupled to bound states, one can study whether a scalar, low mass bound state develops a non-zero vacuum expectation value. These applications are possible if the mean field model is a reasonable approximation of the physics of QCD in the $\vec{p} \to 0$ limit.

Acknowledgement

This research was supported in part by a grant from the National Science Foundation.

Appendix A: Anomalous Interactions

In configuration space $V_1(A)$ is given by[7]

$$V_1(A) = -\frac{g^2}{8}\int d^3r\, d^3r'\, d^3r''\, f_{ade}f_{bec}$$

$$[D_{ab}(\vec{r},\vec{r}';t)\nabla_j'\delta^3(r'-r)][D_{cd}(\vec{r},\vec{r}'';t)\nabla_j''\delta^3(r''-r)] \quad . \tag{A1}$$

In momentum space this becomes

$$\int dt V_1(A) = (2\pi)^4 \frac{g^2}{8}\int d\vec{p}_1\, d\vec{p}_2 d\vec{k}_1\, d\vec{k}_2 d\vec{x}_1\, d\vec{x}_2 \delta^3(p_1+p_2+k_1+k_2)\ \delta(x_1+x_2)$$

$$f_{ade}f_{cbe}\ \vec{k}_1\cdot\vec{k}_2\ D_{ab}(\vec{p}_1,\vec{k}_1;x_1)\ D_{cd}(\vec{p}_2,\vec{k}_2;x_2) \quad . \tag{A2}$$

Although $\int dtV_1(A)$ appears to be of order g^2, the first non-vanishing term is of order g^4. If either factor of the modified Green's function $D_{ab}(\vec{p},\vec{k};x)$ is replaced by its zero order value, $\delta_{ab}\delta(\vec{p}+\vec{k})/p^2$, $\int V_1(A)$ vanishes. When each factor of D_{ab} in (A2) is viewed as a Coulomb line that emits an arbitrary number of gluons, this anomalous term is seen to contain two mutually interacting Coulomb lines which can also exchange gluons with external sources. An important question is whether the interaction between the two lines leads to an enhanced infrared singularity. The singular modified Coulomb interaction $F_{ab}(\vec{p},\vec{k};x)$ is also an integral over a product of modified Green's functions. The color indices and the momentum dependence are different. This problem is addressed in section 6.

The configuration space form of $V_2(A)$ is complicated. In momentum space the expression is marginally simpler.

$$\int dt V_2(A) = -2\pi\frac{g^2N}{8}\int d^3p\, d^3k F_{aa}(\vec{p},-\vec{p};0)\ tr[P(\vec{p})\ P(\vec{p}-\vec{k})]$$

$$-2i(2\pi)^4\frac{g^3}{8}\int d^3p\, d^3k d^3s\, d^3t d^4r\, dx_1 dx_2$$

$$\delta(x_1+x_2+r_o)\ \delta(\vec{p}+\vec{k}+\vec{r}+\vec{s}+\vec{t})\ f_{xey}f_{xfb}f_{yac}$$

$$D_{ab}(\vec{s},\vec{t};x_1)\ F_{ef}(\vec{p},\vec{k};x_2)\vec{A}^c(\vec{r})\cdot P(\vec{s}+\vec{r})\cdot P(\vec{t}+\vec{k})\cdot\vec{t}$$

$$+ \frac{(2\pi)^7 g^4}{8} \int d^3p\, d^3k\, d^3s\, d^3t\, d^3s'\, d^3t'\, d^4r\, d^4r'\, dx_1\, dx_2\, dx_3$$

$$\delta^3(p + t + r' + s')\,\delta^3(k + t' + s + r)\ \delta(x_1 + x_2 + x_3 + r_o + r_o')$$

$$f_{bex}\, f_{fb'y}\, f_{yac}\, f_{xa'c'}\ F_{ef}(\vec{p},\vec{k};x_3)\ D_{ab}(\vec{s},\vec{t};x_1)\ D_{a'b'}(\vec{s}',\vec{t}';x_2)$$

$$(\vec{A}^c(r)\cdot P(\vec{s}+\vec{r})\cdot \vec{t}')(A^{c'}(r')\cdot P(\vec{s}' + \vec{r}')\cdot \vec{t}) \quad , \tag{A3}$$

where $P_{ij}(\vec{p}) = \delta_{ij} - p_i p_j / p^2$. The first term is manifestly ultra-violet divergent. Contributions from $V_2(A)$ are needed to cancel ultra-violet divergences that arise from the Coulomb interaction. Although $F_{ab}(\vec{p},\vec{k};x)$ appears explicitly, the importance of (A3) in the infrared limit is unclear.

Appendix B: Coulomb Operator Product Expansion

When the integral equation for $D_{ab}(\vec{p},\vec{k};x)$ is iterated, the result is an infinite series expansion in powers of the coupling constant g. Substitution of this series into (2.19), the definition of $F_{ab}(\vec{p},\vec{k};x)$, leads to the relation $F = d[gD]/dg$. The structure of the Coulomb interaction operator $F_{ab}(\vec{p},\vec{k};x)$ can be calculated from that of the Green's function operator. The infinite series for $D_{ab}(\vec{p},\vec{k};x)$ is

$$D_{ab}(\vec{p},\vec{k};x) = \frac{1}{(2\pi)^3}\Big[\frac{\delta_{ab}\delta(\vec{p}+\vec{k})\delta(x)}{\vec{p}^{\,2}} + ig\, f_{acb}\frac{1}{\vec{p}^{\,2}}\vec{A}^c(\vec{p}+\vec{k},x)\cdot\vec{p}\,\frac{1}{\vec{k}^{\,2}}$$
$$+ \sum_{n=2}^{\infty}(ig)^n f_{ac_1e_1}f_{e_1c_2e_2}\cdots f_{e_{n-1}c_nb}$$
$$\int\frac{d^4s_1\,\ldots .\,d^4s_n}{\vec{p}^{\,2}(\vec{p}-\vec{s}_1)^2\ldots(\vec{p}-\sum_1^n\vec{s}_i)^2}\,\delta^3(\vec{p}+\vec{k}-\sum_1^n\vec{s}_i)\;\delta(x-\Sigma s_{io}) \tag{B1}$$
$$\vec{A}^{c_1}(s_1)\cdot\vec{p}\,\vec{A}^{c_2}(s_2)\cdot(\vec{p}-\vec{s}_1)\,\ldots\vec{A}^{c_n}(s_n)\cdot(\vec{p}-\sum_1^{n-1}\vec{s}_i).$$

The nth term in this series is comprised of n+1 zeroth order Coulomb propagators, $1/\vec{p}^{\,2}$, separated by n vertices at which gluon fields are coupled. Momentum is conserved at each vertex. If $D_{ab}(\vec{p},\vec{k};x)$ is a sub-unit in a Feynman diagram, some of the gluon fields attach to other elements of the diagram, and the rest become gluons that are emitted and reabsorbed by the Coulomb line itself. (See Figure 1) An alternative expansion of $D_{ab}(\vec{p},\vec{k};x)$ is

$$D_{ab}(\vec{p},\vec{k};x) = \langle D_{ab}(\vec{p},\vec{k};x)\rangle$$
$$+\int d^4s\,D^{(1)}_{acb;i}(\vec{p},\vec{k};x;s)\;A_i^{\,c}(s)\delta^3(p+k-s)\;\delta(x-s_o)$$
$$+\sum_{n=2}^{\infty}\int d^4s_1\ldots d^4s_n\;D^{(n)}_{ac_1\ldots c_nb;\,i_1\ldots i_n}(\vec{p},\vec{k};x;s_1\ldots s_n) \tag{B2}$$
$$:A_{i_1}^{\,c_1}(s_1)\ldots A_{i_n}^{\,c_n}(s_n):\delta^3(p+k-\Sigma s_i)\;\delta(x-\Sigma s_{io})$$

where the : : notation indicates that the gluon fields are to connect to external (i.e. not the same Coulomb line) vertices. Thus for n=2, the contraction between the two field operators is zero. Equation (B2) constitutes an operator product expansion for $D_{ab}(\vec{p},\vec{k};x)$.

The first term in (B2) is the VEV of the modified Coulomb Green's function. One can either calculate it in perturbation theory or use invariance arguments to show that

$$\langle D_{ab}(\vec{p},\vec{k};x)\rangle = \frac{\delta_{ab}}{(2\pi)^3}\,\delta(\vec{p}+\vec{k})\;\delta(x)\,D(\vec{p}) \quad . \tag{B3}$$

The one gluon function $D^{(1)}{}_{abc;i}(\vec{p},\vec{k};x;s)$ has the structure of Figure 6a. There are two factors of the VEV of D_{ab}, and the gluon is emitted from a vertex function which in lowest order is proportional to igp_i.

$$D^{(1)}_{acb;i}(\vec{p},\vec{k};x) = \frac{1}{(2\pi)^3}\,D(p)\{igf_{acb}[p_i + \Gamma_i(p,k)]\}D(k) \quad . \tag{B4}$$

The vertex function $\Gamma_i(\vec{p},\vec{k})$ is one line irreducible. There exists an integral equation for irreducible vertex functions. (See Appendix E) The two gluon function (Figure 6b) is more complicated, but generalization to n gluons is straightforward, at least in principle.

The utility of the operator expansion is that the one, two, etc. gluon vertex functions are benign. They are ultraviolet finite and do not change the nature of the infrared singularity. In Appendix E it is shown that each vertex represents a finite, calculable correction to a lowest order result.

If the vertex functions for gluon emission along a Coulomb line are replaced by the momentum factor of lowest order perturbation theory, the operator expansion of the modified Green's function has exactly the structure of (B1) with each $1/p^2$ replaced by $D(\vec{p})$. Gluons couple to external vertices, not to other gluons along the line. The approximation is justified by the goal of identifying the dominant contributions in the zero momentum limit.

The operator of interest is $F_{ab} = d[gD_{ab}]/dg$. When the nth term of (B1), with $1/\vec{p}^{\,2} \to D(\vec{p})$, is multiplied by g, there are n+1 factors of $D(\vec{p}_i)$ and n+1 powers of g. Thus, the operator expansion for the Coulomb interaction is

$$
\begin{aligned}
F_{ab}(\vec{p},\vec{k};x) = \frac{1}{(2\pi)^3} \{ & \delta_{ab}\delta(\vec{p}+\vec{k})\delta(x)\ F(\vec{p}) \\
& + igf_{acb}[F(\vec{p})\ D(\vec{k}) + D(\vec{p})\ F(\vec{k})]\vec{A}^c(\vec{p}+\vec{k};x)\cdot\vec{p} \\
+ (ig)^2\ & f_{ac_1e_1} f_{e_1c_2b} \int d^4s [F(\vec{p})D(\vec{p}-\vec{s})D(\vec{k}) + D(\vec{p})F(\vec{p}-\vec{s})D(\vec{k}) \\
& + D(\vec{p})\ D(\vec{p}-\vec{s})F(\vec{k})] :\vec{A}^{e_1}(s)\cdot\vec{p}\ \vec{A}^{e_2}(\vec{p}+\vec{k}-\vec{s};x-s_o)\cdot(\vec{p}-\vec{s}): \\
+ \sum_{n=2}^{\infty} & (i)^n f_{ac_1e_1} \ldots f_{e_{n-1}c_nb} \int d^4s_1 \ldots d^4s_n \\
& :\vec{A}^{c_1}(s_1)\cdot\vec{p} \ldots \vec{A}^{c_n}(s_n)\cdot(\vec{p} - \sum_1^{n-1} \vec{s}_i): \\
\frac{d}{dg}\ & [g^{n+1}D(\vec{p})\ D(\vec{p}-\vec{s}_1)\ldots D(\vec{p} - \sum_1^n \vec{s}_n)]\ \delta(\vec{p}+\vec{k}-\sum_1^n \vec{s}_i)\ \delta(x - \sum_1^n s_{io}) \quad .
\end{aligned}
\tag{B5}
$$

The Feynman rules used here are based on this expansion. The general structure of the nth term is a Coulomb line with n+1 segments. One of the segments is infrared singular, $F(\vec{p})$, and others are non-singular, $D(\vec{p})$. The singular Coulomb propagator occurs in n+1 different locations. When external gluons are attached, the modified Coulomb interaction in S_I has the form shown in Figure 6c. The external gluons at either end can be replaced by quarks.

Appendix C: Infrared Singular Limits

Consider an arbitrary Feynman diagram with G internal gluon lines (of any type), Q quark lines, F singular Coulomb lines, and D non-singular Coulomb lines. The number of vertices of each type is specified by n_α, where α indicates the particles coupled. If there are L momentum loops, the amplitude corresponding to a given diagram is of order λ^M where

$$M = F + L - G - Q \quad . \tag{C1}$$

The number of locps is related to the number of lines and the number of vertices of all types.

$$L = 1 + G + Q + F + D - \sum_\alpha n_\alpha \quad . \tag{C2}$$

If there are no external Coulomb lines of either type,

$$2F = n_{ggF} + n_{qqF} + n_{FgD} \quad , \tag{C3a}$$

$$2D = n_{FgD} + 2n_{DgD} + n_{ggD} + n_{qqD} \quad . \tag{C3b}$$

Equations (C2) and (C3) can be used to write M in terms of the number of vertices of each type.

$$M = 1 - n_{gggg} - n_{ggg} - n_{qqg} \quad . \tag{C4}$$

The identity

$$n_{FgD} = n_{ggD} + n_{qqD} \quad , \tag{C5}$$

led to additional cancellations in (C4). This last relation is derived from the observation that a single Coulomb line couples at the ends to either two gluons or two quarks. The Coulomb line is segmented into VEV's of Coulomb operators separated by vertices with gluon emission. (See Appendix B) The linearity of the relationship F = d[gD]/dg means that only one segment along a line is F type and all the rest are D type. If the F segment is at the end of the line, then for that line $n_{FgD} = 1 = n_{ggD} + n_{qqD}$. If the F segment is in the middle, the $n_{FgD} = 2 = n_{ggD} + n_{qqD}$. Thus, (C5) holds for each internal Coulomb line and for a complete diagram with an arbitrary number of Coulomb lines.

The bound on M is actually somewhat stronger in practice. The coupling of quarks and gluons to a singular Coulomb line often vanishes at zero momentum thereby cancelling the λ^{+1} dependence. At an FgD vertex there is an explicit factor of $\vec{p}$. When quarks or gluons are internal, the antisymmetric coupling at ggF and qqF vertices produces a suppression factor at $\vec{p}=0$. Therefore, F in (C1) should be replaced by F_s, the number of singular F lines. The right hand side of (C4) is reduced by $F_s - F$.

Appendix D: Renormalization

Divergences in the integrals of section III are absorbed into seven renormalization constants and the quark mass. The seven constants are defined by $D(\vec{p}) = Z_D D_R(\vec{p})$, $F(\vec{p}) = Z_F F_R(\vec{p})$, $A(\vec{p}) = Z_A A_R(\vec{p})$, $g = Z_g g_R$, $1 + G_1(\vec{p}) = Z_1(1 + G_{1R}(\vec{p}))$, $1 + G_2(\vec{p}) = Z_2(1 + G_{2R}(\vec{p}))$, and $g' = Z_g\, g_R'$. A distinction is made between the gluon-gluon-Coulomb coupling constant g and the quark-quark-Coulomb constant g'. They are equal in lowest order, but their renormalization differs by a finite, calculable factor.

The subtracted and renormalized version of (3.1) is

$$Z_D D_R(\vec{k}) = \frac{1}{k^2}\,[1 - Z_g{}^2 Z_A Z_D\; g_R I_R(\nu)]^{-1} \tag{D1}$$

$$\times\, [1 - \frac{Z_g{}^2 Z_A Z_D}{1 - Z_g{}^2 Z_A Z_D\; g_R I_R(\nu)}\, g_R[I_R(k) - I_R(\nu)]]^{-1} \quad .$$

If

$$Z_D = 1 + g_R I_R(\nu) \quad , \tag{D2}$$

and

$$Z_g{}^2 Z_D{}^2 Z_A = 1 \quad , \tag{D3}$$

equation (D1) becomes

$$g_R D_R(\vec{k}) = \frac{g_R}{k^2[1 - g_R[I_R(k) - I_R(\nu)]]} \quad . \tag{D4}$$

When the running coupling constant $g(k) = k^2 g_R D_R(k)$ is introduced, (D4) is equivalent to

$$\frac{1}{g(k)} = \frac{1}{g_R} - [I_R(k) - I_R(\nu)] \quad , \tag{D5}$$

and $g_R = g(\nu)$. Equation (3.25c) is the result of subtracting (D5) once more at $\vec{k} = 0$.

The renormalization of (3.4) for F(k) proceeds in the same way.

$$Z_F F_R(k) = k^2 D(k)^2 [1 + g^2 J(k)] = \frac{Z_D{}^2}{k^2}\,\frac{g(k)^2}{g_R{}^2}\,[1 + Z_g{}^2 Z_A Z_F\; g_R{}^2 J_R(\nu)] \tag{D6}$$

$$\times\, [1 + \frac{Z_g{}^2 Z_A Z_F}{1 + Z_g{}^2 Z_A Z_F\; g_R{}^2 J_R(\nu)}\, g_R{}^2[J_R(k) - J_R(\nu)]] \quad .$$

The choice

$$Z_F = Z_D^{\,2}[1 + Z_g^{\,2} Z_A Z_F\, g_R^{\,2} J_R(\nu)] \quad , \tag{D7}$$

converts (D6) into (49b).

Two subtractions are needed to render finite the integrals in (3.15). This is accomplished by writing (3.15) in the form

$$\frac{1 + \bar{F}_1(\nu) + [\bar{F}_1(k) - \bar{F}_1(\nu)]}{A(k)^2} - \frac{1 + \bar{F}_1(\nu) + [\bar{F}_1(k) - \bar{F}_1(p)]}{A(p)^2}$$
$$= (k^2 - p^2)[1 + \frac{\bar{F}_2(\nu) - \bar{F}_2(p)}{\nu^2 - p^2}] \tag{D8}$$
$$+ \bar{F}_2(k) - \bar{F}_2(p) - \frac{(k^2 - p^2)}{(\nu^2 - p^2)}[\bar{F}_2(\nu) - \bar{F}_2(p)] \quad .$$

Next renormalization constants are introduced and the subtraction momentum p is set equal to zero.

$$\frac{1 + Z_g^{\,2} Z_F Z_A\, \bar{F}_{1R}(\nu)}{Z_A^{\,2}}\left[\frac{1 + f_1[\bar{F}_{1R}(k) - \bar{F}_{1R}(\nu)]}{A(k)^2} - \frac{1 + f_1[\bar{F}_{1R}(0) - \bar{F}_{1R}(\nu)]}{A(0)^2}\right]$$
$$= (1 + \frac{Z_g^{\,2} Z_F}{Z_A}\frac{\bar{F}_{2R}(\nu) - \bar{F}_{2R}(0)}{\nu^2})[k^2 + f_2(\bar{F}_{2R}(k) - \bar{F}_{2R}(0) - \frac{k^2}{\nu^2}(\bar{F}_{2R}(\nu) - \bar{F}_{2R}(0))], \tag{D9}$$

where

$$f_1 = \frac{Z_g^{\,2} Z_F Z_A}{1 + Z_g^{\,2} Z_F Z_A\, \bar{F}_{1R}(\nu)} \quad , \tag{D10a}$$

$$f_2 = \frac{Z_g^{\,2} Z_F / Z_A}{1 + \frac{Z_g^{\,2} Z_F}{Z_A}\frac{\bar{F}_{2R}(\nu) - \bar{F}_{2R}(0)}{\nu^2}} \quad . \tag{D10b}$$

Using the conditions

$$Z_A^{\,2} = \frac{1 + Z_g^{\,2} Z_F Z_A\, \bar{F}_{1R}(\nu)}{1 + \frac{Z_g^{\,2} Z_F}{Z_A}\frac{\bar{F}_{2R}(\nu) - \bar{F}_{2R}(0)}{\nu^2}} \quad , \tag{D11}$$

and

$$1 = \frac{Z_g^{\,2}\, Z_F\, Z_A}{1 + Z_g^{\,2} Z_F Z_A\, \bar{F}_{1R}(\nu)} \quad , \tag{D12}$$

one finds $f_1 = f_2 = 1$, and (D9) becomes (3.25c). Equation (D12) is compatible with (D2), (D3), and (D7) if the divergent parts of $J_R(\nu)$ and $F_{1R}(\nu)$ are equal. This condition on the renormalizability of the mean field model is satisfied. The coupling constants at the DgD and ggF vertices remain equal after renormalization. In addition, Z_A is finite.

Quark counterterms are renormalized in a similar way. Equation (3.19) takes on the form

$$\begin{aligned} Z_1[1 + \bar{G}_{1R}(k)] &= [1 + Z_F Z_{g'}^{\,2} IG_{1R}(\nu)] \\ &\quad \times [1 + g_1[IG_{1R}(k) - IG_{1R}(\nu)]] \quad , \end{aligned} \tag{D13a}$$

$$\begin{aligned} Z_2[1 + \bar{G}_{2R}(k)] &= [1 + \frac{Z_F Z_{g'}^{\,2} Z_2}{Z_1} IG_{2R}(\nu)] \\ &\quad \times [1 + g_2[I\bar{G}_{2R}(k) - I\bar{G}_{2R}(\nu)]] \quad , \end{aligned} \tag{D13b}$$

where

$$g_1 = \frac{Z_F\, Z_{g'}^{\,2}}{1 + Z_F Z_{g'}^{\,2}\; IG_{1R}(\nu)} \quad , \tag{D14a}$$

$$g_2 = \frac{Z_F Z_{g'}^{\,2}\; Z_2/Z_1}{1 + \dfrac{Z_F Z_{g'}^{\,2} Z_2}{Z_1}\; IG_{2R}(\nu)} \quad , \tag{D14b}$$

and $IG_i(k)$ stands for the integrals on the right hand side of (3.19). One of the factors in the integrand is the infrared finite quark energy function

$$\bar{E}(s) = Z_1[s^2(1 + \bar{G}_{1R})^2 + m_R^{\,2}(1 + \bar{G}_{2R})^2]^{1/2} = Z_1 \bar{E}_R(s) \quad . \tag{D15}$$

The renormalized quark mass is $m_R = (Z_1/Z_2)m = (4/3)m$. If

$$Z_1 = 1 + Z_F Z_{g'}^{\,2}\; IG_{1R}(\nu) \quad , \tag{D16a}$$

$$Z_2 = 1 + \frac{Z_F Z_{g'}^{\,2} Z_2}{Z_1}\; IG_{2R}(\nu) \quad , \tag{D16b}$$

$$Z_F Z_{g'}^{\,2} = Z_1 \quad , \tag{D16c}$$

all divergences cancel, $g_1 = g_2 = 1$, and (D13) becomes (3.28).

The divergences in the various integrals can be analyzed with dimensional regularization. Poles appear at $d = 3$ dimensions. Certain ratios of renormalization constants are finite. In particular

$$Z_g^{\;2}/Z_{g'}^{\;2} = \frac{1 - IG_{1R}(\nu)}{\left[(1 - F_{1R}(\nu))\left(1 - \frac{(F_{2R}(\nu) - F_{2R}(0))}{\nu^2}\right]^{1/2}\right.} = \left(\frac{5}{3}\right)^{1/2} \frac{N^2-1}{2N^2} \quad . \qquad \text{(D17)}$$

The quark and gluon couplings to a Coulomb line differ by a finite factor after renormalization, a consequence of the suppression of certain diagrams. (When $N = 3$, $Z_g^{\;2}/Z_{g'}^{\;2} = .57$). In order to recover the results of conventional QCD renormalization, one must hold the divergent constant $\lambda(\mu)$ fixed both for the p_o integrations and also for renormalization. Only after renormalization should the $\lambda \to \infty$ limit be taken. The missing diagrams are not expected to make finite contributions to any amplitudes.

Appendix E: Corrections

Perturbation theory in the mean field model is simplified by the elimination of quark and gluon self energies and the suppression of certain vertices. The price of that simplification is a non-local Coulomb interaction. The derivation of the self consistency equations required, as a practical matter, a number of approximations. In particular calculations in the effective field theory were carried out to order g^2. This appendix is devoted to a discussion of a variety of higher order diagrams. If new infrared singularities appear or if ones present in lowest order disappear, the model is not stable under perturbations. The diagrams to be discussed appear in Figure 7. The question of convergence of the perturbation series is not addressed.

The operator product expansions for the Green's function and the modified Coulomb interaction were simplified by the neglect of vertex corrections at the points where gluons are emitted from a Coulomb line. Figure 7a shows the first correction to a point vertex. If a gluon of momentum $\vec{p}+\vec{k}$ is emitted by a D-line carrying momentum $\vec{p}$, the factor $\vec{A}(\vec{p}+\vec{k})\cdot\vec{p}$ is replaced by $\vec{A}(\vec{p}+\vec{k})\cdot[\vec{p} + \vec{\Gamma}(\vec{p},\vec{k})]$ where

$$\Gamma_i(\vec{p},\vec{k}) = -\alpha \int d^3s \; D(\vec{p}-\vec{s}) \; D(-\vec{k}-\vec{s}) \; \vec{p}\cdot P(\vec{s})\cdot\vec{k} \quad \vec{A}(s)[\vec{p}-\vec{s}]_i \qquad \text{(E1)}$$

The integral is convergent; no ultra-violet renormalization is necessary. There are no infrared singularities. The infrared limit of (E1) is obtained by scaling all momenta by a common factor γ and using the infrared limits of $D(s) \propto s^{-5/2}$ and $A(s) \propto s^0$. One finds that both the point vertex term p_i and Γ_i are of order γ. Hence, the use of (E1) in the operator product expansion makes no qualitative change in infrared or ultraviolet properties.

Figure 7b depicts a generic higher order contribution to $\Gamma_i(\vec{p},\vec{k})$. Simple power counting for a diagram with N internal gluons and 2N internal D-lines shows that there are no ultraviolet divergences in any order. The scaling argument indicates that the infrared limits of D(s) and A(s) are matched so that

$\Gamma_i \propto \gamma$ in any order. Moreover, as a vector, $\Gamma_i(\vec{p},\vec{k})$ is proportional to a linear combination of the vectors $\vec{p}$ and $\vec{k}$. In the limit $\vec{p}\to 0$, $\vec{A}(\vec{p}+\vec{k})\cdot\vec{\Gamma}(\vec{p},\vec{k})$ vanishes. There is as much infrared suppression from the full vertex function as there is from a point vertex. Thus, on the assumption that higher order in g^2 means smaller, the neglect of vertex corrections in the operator product expansion is justified.

Smallness of finite higher order corrections can be tested with Figure 7c. One has a vertex insertion in the integral I(k) used in (3.1) and (3.2) to calculate D(k). Numerical evaluation is necessary. Consistent with the scaling behavior of the vertex, there is no change in the $\vec{k}\to 0$ power dependence of D(k). If $D(k) \simeq \eta k^{-5/2}$, then (3.25a) leads to a condition of the form $1 = \eta^2 I_o$. The vertex correction changes this to $1 = \eta^2 I_o + \eta^4 I_1 \simeq \eta^2 I_o[1 + I_1/I_o{}^2] = \eta^2 I_o[1 + 0.11]$. There is a 5% shift in the coefficient η. The derivative of Figure 7c with respect to coupling constant produces corrections in the integral equation for F(k). The small shift in coefficients is not in the direction needed to produce a real power law solution for F(k) at $\vec{k}\to 0$.

Figure 7d illustrates another type of ostensibly infrared finite correction to I(k). There are three loop integrals, four propagators, and a factor of F(k). However, as mentioned in Appendix C, the antisymmetry of the coupling of F(k) to $A_i{}^a$ and $P_i{}^a$ suppresses the singularity when there are no external gluons attached to the Coulomb propagator. In that case there is a sum over the different types of gluon propagators. Figure 7d is of order λ^{-1} and vanishes in the infrared limit.

The full operator product expansion for $F_{ab}(\vec{p},\vec{k};x)$ was not used in the calculation of the counterterm functions. Two gluon terms in the expansion produce the g^4 corrections to the gluon self energy shown in Figures 7e,f,g and h. Unlike diagrams with three gluon, four gluon, or quark-quark-gluon vertices, these contributions do not vanish in the infrared limit. The gluonic corrections to $F_1(k)$ and $F_2(k)$ are

$$F_1^{(4)}(k) = \frac{\alpha^2}{4}\int d^3s\, d^3t\, A(s)A(t)[(\vec{k}-\vec{s})\cdot P(\vec{t})\cdot P(\vec{k})\cdot P(\vec{s})\cdot(\vec{k}+\vec{t})]$$

$$\frac{1}{g^2}\frac{d}{dg}[g^3\, D(\vec{k}+\vec{t})\, D(\vec{k}+\vec{t}-\vec{s})\, D(\vec{k}-\vec{s})] \quad , \tag{E2}$$

and

$$F_2^{(4)}(k) = \frac{\alpha^2}{2}\int d^3s\, d^3t\, [A(s)/A(t) -1]\, tr[P(\vec{s})\cdot P(\vec{t})]$$

$$[(\vec{s}+\vec{t})\cdot P(\vec{k})\cdot(\vec{s}+\vec{t})]\, \frac{1}{g^2}\frac{d}{dg}[g^3\, D(\vec{s}+\vec{t})^2\, D(\vec{s}+\vec{t}+\vec{k})] \tag{E3}$$

$$+ \alpha^2 \int d^3s\, d^3t\, \frac{A(s)}{A(t)}\, [(\vec{t}-\vec{k})\cdot P(\vec{s})\cdot P(\vec{t})\cdot P(\vec{k})\cdot(\vec{s}+\vec{t})]$$

$$\frac{1}{g^2}\frac{d}{dg}[g^3\, D(\vec{k}-\vec{t})\, D(\vec{k}-\vec{t}-\vec{s})\, D(-\vec{t}-\vec{s})] \quad .$$

The quark loop diagram of Figure 7h adds to (E3) but is suppressed by a power of N. The coupling constant derivative generates the singular function F(k) through $F = d[gD]/dg$. In each integral there are factors that eliminate the delta function in $F(\vec{k}) \simeq \lambda\delta^3(k)$, and there are no manifest infrared singularities in (E2) or (E3). For comparison the lowest order terms in (3.11) are singular. A scaling argument indicates that (E2) and (E3) have the same $k \to 0$ behavior as (3.11). Hence, the higher order terms in the operator product expansion of $F_{ab}(\vec{p},\vec{k};x)$ do not generate singularities but rather produce finite, higher order corrections.

The quark loop in Figure 7i was dropped because it vanishes in the infrared limit. Its contribution to $F_2(k)$ is

$$F_2'(k) = \frac{2g^2}{(2\pi)^3}\int \frac{d^3s\, d^3t\, \delta^3(s+t-k)}{\bar{E}(s)\, \bar{E}(t)}\, \frac{2\Sigma}{\Sigma^2 - k_o^2}$$

$$\times\, [\bar{E}(s)\, \bar{E}(t) + m^2(1+\bar{G}_2(s))(1+\bar{G}_2(t)) \tag{E4}$$

$$+ (\vec{s}\cdot\vec{t} + \vec{s}\cdot P(\vec{k})\cdot\vec{t})(1+\bar{G}_1(s))(1+\bar{G}_1(t))] \quad ,$$

where $\Sigma = 2\alpha'\lambda + \bar{E}(s) + \bar{E}(t)$. This integral is of order λ^{-1}, but it is also quadratically divergent. Using dimensional regularization, one finds that the coefficient of the pole is of order λ^2. Moreover, $F_2'(k)$ is a function of k_o^2, contrary to an essential assumption. However, the renormalization procedure used in (3.24) involves subtractions at $k^2=0$ and $k^2=\nu^2$. The residual finite integral vanishes as $\lambda \to \infty$. There is no contribution from the quark loop, and $F_2(k)$ is a function of three-momentum only.

The last corrections to be considered here are shown in Figure 7j,k. These are quark and gluon loop insertions in the singular Coulomb propagator. Although the VEV of the Coulomb interaction operator is infrared singular, quarks and gluons actually interact via the Coulomb interaction as modified by these self energy insertions. The singular function F(k) should be replaced by F'(k) in the counterterm integrals of section 3 and in the Bethe-Salpeter equations of section V. The two functions are related by

$$F'(k)^{-1} = F(k)^{-1} + I_G(k) + I_Q(k) \quad . \tag{E5}$$

The gluon loop integral is

$$I_G(k) = -\frac{\alpha}{2}\int d^3s\, d^3t\, \delta^3(s+t-k)\; tr[P(\vec{s})P(\vec{t})]\left[\frac{A(s)}{A(t)} + \frac{A(t)}{A(s)} - 2\right] \frac{2\alpha\lambda + \bar{\omega}(s) + \bar{\omega}(t)}{[(2\alpha\lambda + \bar{\omega}(s) + \bar{\omega}(t))^2 - k_o^2]} \quad , \tag{E6}$$

and the quark term is

$$I_Q(k) = \frac{\alpha}{N}\int d^3s\, d^3t\, \frac{\delta^3(s+t-k)}{\bar{E}(s)\,\bar{E}(t)} \frac{2\alpha'\lambda + \bar{E}(s) + \bar{E}(t)}{[(2\alpha'\lambda + \bar{E}(s) + \bar{E}(t))^2 - k_o^2]} [m^2(1+\bar{G}_2(s))(1+\bar{G}_2(t)) - \vec{s}\cdot\vec{t}(1+\bar{G}_1(s))(1+\bar{G}_1(t)) - \bar{E}(s)\,\bar{E}(t)] \quad . \tag{E7}$$

Both integrals are logarithmically divergent, functions of k_o^2, and of order λ^{-1}. They are also proportional to k^2 in the $k\to 0$ limit. If $\lambda \to \infty$, both integrals vanish. Closer analysis reveals a problem. If one considers the dependence on the cut-off parameter μ, one discovers that I_G and I_Q are proportional to $\mu^{2n-3}k^2$. $[\lambda^{-1} \propto \mu^{2n-3}]$ Therefore, if $F(k)^{-1} \propto [k^2+\mu^2]^n$,

$$F'(k)^{-1} = [k^2+\mu^2]^n + \beta\mu^{2n-3}k^2 \quad , \tag{E8}$$

where β is linear in the quark mass and $1/A(0)$. Integrals with F(k) replaced by $F'(k)$ diverge as $\mu \to 0$, but the actual μ dependence is not consistent with $\lambda \propto \mu^{3-2n}$. In other words, if (E8) is true, the infrared singularity in $F(k)$ is not maintained in higher order.

When (E5) is renormalized, the problem goes away. The quark and gluon integrals are both proportional to k^2. If $F(k) = f(k)/k^2$, $F'(k) = f'(k)/k^2$, and $I_G(k) + I_Q(k) = k^2 I(k)$,

$$f'(k)^{-1} - f'(\nu)^{-1} = f(k)^{-1} - f(\nu)^{-1} + I(k) - I(\nu) \quad . \qquad \text{(E9)}$$

The subtraction renders the integrals ultraviolet finite. Choosing $\nu = 0$ and requiring $f'(0)^{-1} = f(0)^{-1} = 0$, one finds

$$f'(k)^{-1} = f(k)^{-1} + I(k) - I(0) \quad . \qquad \text{(E10)}$$

Since $I(k) - I(0)$ is proportional to k^2 as $k \to 0$, the infrared singularity in $f(k)$ is maintained in $f'(k)$. One can safely let $\mu \to 0$, $\lambda \to \infty$, and $[I(k) - I(0)] \to 0$. While it might seem that an obvious point has been belabored, the existence of the infrared singularity in $F'(k)$ as well as in $F(k)$ is absolutely crucial for the whole model. Were a higher order correction to modify the $k \to 0$ limit, the mean field model would collapse.

Appendix F: Bethe-Salpeter Equation for D-Type Coulomb Lines

The modified Coulomb interaction is defined in terms of a particular integral (2.19) of the product of two modified Coulomb Green's functions. Direct calculation of the VEV of the interaction requires the ability to calculate the VEV of the product of two Green's functions. In addition, to understand the anomalous interactions, one must compute the VEV of products of Green's functions and/or modified interactions. Consider the quantity

$$\delta(z)H^{ab}_{cd}(\vec{p},\vec{k};\vec{q},\vec{r}) = (2\pi)^6 \int dxdy\ \delta(x+y-z)\langle D_{ab}(\vec{p},\vec{k};x)\ D_{cd}(\vec{q},\vec{r};y)\rangle . \tag{F1}$$

When the operator product expansion of Appendix B (with point vertices) is used for $D_{ab}(\vec{p},\vec{k};x)$, one is faced with summing the diagrams of Figure 8. The two Coulomb lines interact with each other via gluon exchange. What is needed is the Bethe-Salpeter equation for Green's function - Green's function scattering. In the ladder approximation the two contributing amplitudes are related by crossing.

$$\begin{aligned}\delta(z)H^{ab}_{cd}(\vec{p},\vec{k};\vec{q},\vec{r}) &= \delta_{ab}\delta_{cd}\delta(\vec{p}+\vec{k})\ \delta(\vec{q}+\vec{r})\ D(\vec{p})\ D(\vec{q})\ \delta(z) \\ &+ \int dxdy\ \delta(x+y-z)\{\Phi^{ab}_{cd}(\vec{p},\vec{q};\vec{k},\vec{r};x,y) \\ &\quad + \Phi^{ab}_{dc}(\vec{p},\vec{r};\vec{k},\vec{q};x,y) - \Phi_{o}{}^{ab}_{cd}(\vec{p},\vec{q};\vec{k},\vec{r};x,y)\}\ .\end{aligned} \tag{F2}$$

The single gluon exchange amplitude is

$$\begin{aligned}&\int dxdy\ \delta(x+y-z)\ \Phi_{o}{}^{ab}_{cd}(\vec{p},\vec{k};\vec{q},\vec{r};x,y) \\ &= -\delta(z)\ \frac{\alpha}{N}\ f_{aeb}f_{ced}\ D(\vec{p})D(\vec{k})D(\vec{q})D(\vec{r})\ \vec{p}\cdot P(\vec{p}+\vec{k})\cdot\vec{q}A(\vec{p}+\vec{k}) \\ &\quad \times\ \delta(\vec{p}+\vec{k}+\vec{q}+\vec{r})\ .\end{aligned} \tag{F3}$$

Since Φ_o is common to both ladder amplitudes, it is subtracted to avoid double counting.

A general ladder diagram has both colored and color singlet components. Only the color singlet one is expected to produce an enhanced infrared singularity. When the singlet part is projected out,

$$\int dx\,dy\ \delta(x+y-z)\ \Phi^{ab}_{cd}\ (\vec{p},\vec{q};\vec{k},\vec{r};\vec{x},\vec{y})$$

$$= \frac{\delta_{ac}\delta_{bd}}{N^2-1}\ \delta(z)\ D(\vec{p})D(\vec{q})D(\vec{k})D(\vec{r})\ \delta(\vec{p}+\vec{k}+\vec{q}+\vec{r})\ \Psi(\frac{\vec{p}-\vec{q}}{2}, -\frac{(\vec{k}-\vec{r})}{2}; \frac{\vec{p}+\vec{q}}{2}) \quad , \tag{F4}$$

and $\Psi(\vec{s},\vec{t};\vec{E})$ is a three dimensional Bethe-Salpeter wave function. The integral equation is

$$\Psi(\vec{s},\vec{t};\vec{E}) = \alpha A(\vec{s}-\vec{t})(\vec{s}-\vec{E})\cdot P(\vec{s}-\vec{t})\cdot(\vec{s}+\vec{E}) + \alpha\int d^3u\ D(\vec{E}+\vec{u})D(\vec{E}-\vec{u})(\vec{s}-\vec{E})\cdot P(\vec{s}-\vec{u})\cdot(\vec{s}+\vec{E})\ A(\vec{s}-\vec{u})\ \Psi(\vec{u},\vec{t};\vec{E}). \tag{F5}$$

The VEV of a product of two Green's functions is

$$\begin{aligned} H^{ab}_{cd}(\vec{p},\vec{k};\vec{q},\vec{r}) &= \delta_{ab}\delta_{cd}\ \delta(\vec{p}+\vec{k})\ \delta(\vec{p}+\vec{k})\ \delta(\vec{q}+\vec{r})\ D(\vec{p})\ D(\vec{q}) \\ &+ \frac{D(\vec{p})D(\vec{q})D(\vec{k})D(\vec{r})}{N^2-1}\ \delta(\vec{p}+\vec{k}+\vec{q}+\vec{r})\{\delta_{ac}\delta_{bd}[\Psi(\frac{\vec{p}-\vec{q}}{2}, \frac{\vec{r}-\vec{k}}{2}; \frac{\vec{p}+\vec{k}}{2}) + \alpha\vec{p}\cdot P(\vec{p}+\vec{k})\cdot\vec{q}A(\vec{p}+\vec{k})] \\ &+ \delta_{ad}\delta_{bc}[\Psi(\frac{\vec{p}-\vec{r}}{2}, \frac{\vec{q}-\vec{k}}{2}; \frac{\vec{p}+\vec{r}}{2}) - \alpha\vec{p}\cdot P(\vec{p}+\vec{k})\cdot\vec{q}A(\vec{p}+\vec{k})] \\ &- \alpha\frac{(N^2-1)}{N}\ f_{aeb}f_{ced}\ \vec{p}\cdot P(\vec{p}+\vec{k})\cdot\vec{q}A(\vec{p}+\vec{k})\} \quad . \end{aligned} \tag{F6}$$

The VEV of $F_{ab}(\vec{p},\vec{k};z)$ is proportional to the infrared singular function $F(\vec{k})$, where

$$\begin{aligned} F(\vec{k})\ \delta(\vec{p}+\vec{k})\delta_{ad} &= \int d^3s\ s^2\ H^{ab}_{bd}(\vec{p},\vec{s};-\vec{s},\vec{k}) \\ &= \delta_{ad}\delta(\vec{p}+\vec{k})\ D(\vec{k})^2 \\ &\{k^2 + \int d^3s\,D(\vec{s})^2[\frac{1}{N^2-1}\ \Psi(\frac{-\vec{k}+\vec{s}}{2}, \frac{\vec{k}-\vec{s}}{2}; -\frac{(\vec{k}+\vec{s})}{2}) \\ &+ \Psi(-\vec{k},-\vec{s};0) - \frac{1}{N^2-1}\ \alpha\ \vec{k}\cdot P(\vec{s}-\vec{k})\cdot\vec{k}\ A(\vec{s}-\vec{k})]\} \quad . \end{aligned} \tag{F7}$$

Retaining just those terms which survive the $N\to\infty$ limit, one finds

$$F(\vec{k}) = D(\vec{k})^2[k^2 + \Psi(\vec{k})] \tag{F8}$$

where

$$\Psi(\vec{k}) \equiv \int d^3s\ D(\vec{s})^2\ \Psi(-\vec{k},-\vec{s};0)$$

$$= \alpha \int d^3s\ D(\vec{s})^2\ s^2 \vec{k}\cdot P(\vec{s}-\vec{k})\cdot\vec{k}\ A(\vec{s}-\vec{k}) \tag{F9}$$

$$+ \alpha \int d^3u\ D(u)^2\ \vec{k}\cdot P(\vec{k}-\vec{u})\cdot\vec{k}\ A(\vec{u}-\vec{k})\ \Psi(\vec{u}) \quad .$$

When (F8) is used to write $\Psi(\vec{k})$ in terms of $F(\vec{k})$, the integral equation for $\Psi(\vec{k})$ becomes (3.4), the integral equation for $F(k)$.

This alternate derivation of (3.4) is useful because it suggests that the singularity in $F(\vec{k})$ arises from the binding of two D-type Coulomb lines by gluon exchange. Since there is no need to compute the derivative with respect to coupling constant of unknown functions, it is easier in this approach to identify possible corrections to $F(\vec{k})$. Approximations that could be improved are the ladder approximation, the restriction to color singlet configurations, and the neglect of the crossed ladder amplitude.

Appendix G

Finite energy bound states constitute a new class of particles that can be emitted, absorbed, and propagated. Moreover, in a confining theory there are an infinite number of such states. The set of Feynman rules for gluons and quarks should be augmented by rules for bound state interactions. A complete treatment is beyond the scope of this paper. However, a brief discussion of bound state amplitudes provides a nice illustration of the power and completeness of the mean field model.

There is a standard procedure for normalizing bound state Bethe-Salpeter wave functions.[27] Solutions to the inhomogeneous equation develop a pole as the time component of the total center of momentum four vector approaches the bound state energy. The residue of that pole is an outer product of the bound state wave functions. Normalization of the residue is tied to the strength of the inhomogeneous term in the Bethe-Salpeter wave function. The gluon integral equation is given in (5.10) with the inhomogeneous term of (5.5). The wave function $\Psi(\vec{p},\vec{k};\vec{W})$ in the region $W_o \simeq E$ has the form

$$\Psi^{ir}_{kt}(\vec{p},\vec{k};\vec{W},W_o)^{js}_{mu} = \frac{-i}{8\pi}\frac{1}{W_o - E}\,\psi^{im}_{kt}(\vec{p},\vec{W})\;\psi^{\dagger js}_{mu}(\vec{k},\vec{W}), \tag{G1}$$

where

$$\begin{aligned}\psi^{ir}_{kt}(\vec{p},\vec{W}) = \theta_{rr'}(\tfrac{\vec{W}}{2}+\vec{p})\theta_{tt'}(\tfrac{\vec{W}}{2}-\vec{p})\\ [(\Sigma(\vec{p},\vec{W}) + E)\chi^{(+)}_{r'}\;\chi^{(+)}_{t'}\;\Theta_{+}{}^{i}_{k}(\vec{p},\vec{W})\\ + (\Sigma(\vec{p},\vec{W}) - E)\chi^{(-)}_{r'}\;\chi^{(-)}_{t'}\;\Theta_{-}{}^{i}_{k}(\vec{p},\vec{W})]\end{aligned} \quad , \tag{G2}$$

and $\Sigma(\vec{p},\vec{W}) = 2\alpha\lambda + \bar{\Sigma}(\vec{p},\vec{W})$. The wave functions $\Theta_+(\vec{p},\vec{W})$ and $\Theta_-(\vec{p},\vec{W})$ are solutions of (5.19) with $W_o = E$. The normalization condition is

$$\begin{aligned}1 = \int d^3p\; A(\tfrac{\vec{W}}{2}+\vec{p})A(\tfrac{\vec{W}}{2}-\vec{p})[\Theta_{+}^{*}{}^{i}_{k}(\vec{p},\vec{W})\;\Theta_{+k}^{\;i}(\vec{p},\vec{W})\\ - \Theta_{-}^{*}{}^{i}_{k}(\vec{p},\vec{W})\;\Theta_{-k}^{\;i}(\vec{p},\vec{W})]\end{aligned} \quad . \tag{G3}$$

There is no dependence on the infrared cut-off.

The effective Hamiltonian for four gluons interacting via a color singlet bound state is

$$H_{eff} = - \frac{(2\pi)^8}{64\pi} \int d^4W d^4p\, d^4k\, \frac{\delta_{bd}\delta_{ce}}{N^2-1}$$

$$A^b_{ir}(W/2 + p)\, A^c_{js}(- \frac{W}{2} - k)\, A^d_{kt}(\frac{W}{2} - p)\, A^e_{mu}(- \frac{W}{2} + k) \qquad \text{(G4)}$$

$$\times \frac{\psi^{ir}_{kt}(\vec{p},\vec{W})\; \psi^{\dagger}{}^{js}_{mu}(\vec{k},\vec{W})}{W_o - E} \;.$$

Each pole at $W_o = E$ is matched by a pole at $W_o = -E$. One notices that (5.19) is invariant under $W_o \to -W_o$ and $\Theta_+ \leftrightarrow \Theta_-$. Thus, the residue of the pole at $-E$ is given by (G2) with $E \to -E$ and $\Theta_+ \leftrightarrow \Theta_-$. Although the wave functions Θ_+ and Θ_- are infrared finite, each residue factor $\psi^{in}_{kt}(\vec{p},\vec{W})$ in (G4) contains $\Sigma(\vec{p},\vec{W}) = 2\alpha\lambda + \bar{\Sigma}(\vec{p},\vec{W})$. Thus, the effective interaction is of order λ^2. It appears that (G4) dominates all interactions. However, a bound state couples through gluon propagators, each of which introduces suppression factors. In addition, loop integrations which encircle gluon poles force $W_o \propto \lambda$ in the denominator of (G4). Thus, if (G4) replaces the Coulomb interaction of Figure 7d of the correction diagrams considered in Appendix E, there are three loop integrations, four gluon propagators, and a bound state propagator to produce a g^4 correction of λ^o. In the gluon self energy one finds a λ^1 term. Bound state diagrams are no more singular than those already considered. In addition, the strength of bound state diagrams is suppressed by color effects. The usual factor of N is missing and there is a $1/(N^2-1)$ from the color singlet projection operator.

Additional suppression cones from the coupling in type space. There is a rich phenomenology associated with the analysis of bound state interactions. A detailed treatment requires solutions to the bound state equations.

References

1. A. Hasenfaratz in Hadron Spectroscopy - 1985, proceedings of an international conference at the University of Maryland, edited by S. Oneda (AIP, New York, 1985), p. 249.
2. J. Finger, D. Horn, and J. E. Mandula, Phys. Rev. D20, 3253 (1979); V. P. Nair and C. Rosenzweig, Phys. Rev. D31, 401 (1985); C. Thorn, Phys. Rev. D19, 639 (1979); C. Callan, R. Dashen, and D. J. Gross, Phys. Rev. D17, 2717 (1978); T. K. Hansson, K. Johnson, and C. Peterson, Phys. Rev. D26, 2069 (1982); H. Flyvbjerg, Nucl. Phys. B176, 379 (1981).
3. A. R. Swift and J. L. Rodriguez Marrero, Phys. Rev. D29, 1823 (1984).
4. J. L. Rodriguez Marrero and A. R. Swift, Phys. Rev. D31, 917 (1985).
5. J. L. Rodriguez Marrero and A. R. Swift, Phys. Rev. D32, 476 (1983).
6. C. M. Bender, T. Eguchi, and H. Pagels, Phys. Rev. D17, 1086 (1978); R. Jackiw, I. Muzinich, and C. Rebbi, Phys. Rev. D17, 1576 (1978); R. E. Cutkosky, Phys. Rev. D30, 447 (1984); R. E. Peccei, Phys. Rev. D17, 1987 (1978).
7. N. H. Christ and T. D. Lee, Phys. Rev. D22, 939 (1980).
8. T. D. Lee, Particle Physics and Introduction to Field Theory (Harwood Academic Publishers, Chur, Switzerland, 1981) Chapter 19.
9. L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).
10. H. Hamber and G. Parsi, Phys. Rev. Letters 47, 1792 (1981); D. Weingarten, Phys. Lett. 109B, 57 (1982).
11. K. G. Wilson, Phys. Rev. D10, 2445 (1974).
12. V. N. Gribov, Nucl. Phys. B139, 1 (1978); D. Zwanziger, Nucl. Phys. B209, 336 (1982).
13. M. Bando, K. Fujii, Y. Abe, T. Okazaki, and S. Kurada, Phys. Rev. D33, 548 (1986).
14. E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973); F. L. Feinberg, Phys. Rev. D17, 2659 (1978).

15. H. Cheng and E-C. Tsai, Phys. Rev. Letters 57, 511 (1986).

16. R. E. Cutkosky in Hadron Spectroscopy - 1985, proceedings of an international conference at the University of Maryland, edited by S. Oneda (AIP, New York, 1985) page 227.

17. G. 't Hooft, Nucl. Phys. B72, 461 (1974); B75, 461 (1974); E. Witten, Nucl. Phys. B160, 57 (1979).

18. J. Schwinger, Phys. Rev. 127, 324 (1962).

19. L. S. Brown and W. I. Weisberger, Phys. Rev. D20, 3239 (1979).

20. A. R. Swift and J. I. Donna, Phys. Rev. D19, 657 (1979).

21. I. Tamm, J. Phys. (Moscow) 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78 382 (1950); F. Dyson, Phys. Rev. 90, 994 (1953); 91, 1543 (1953); J. C. Taylor, Phys. Rev. 95, 1313 (1953).

22. N. D. Son and J. Sucher, Phys. Rev. 153, 1496 (1967); B. Silvestre-Brac, A. Bilol, C. Gignoux, and P. Schuck, Phys. Rev. D29, 2275 (1984).

23. J. L. Rosner in Proceedings of the 1985 International Symposium on Lepton and Photon Interactions at High Energies, edited by M. Konuma and K. Takahashi (Organizing Committee, 1985 International Symposium, Kyoto, Japan, 1986) page 448.

24. E. Eichten, K. Gottfried, T. Kinoshita, K. Lane, and T. Yan, Phys. Rev. D21, 203 (1980).

25. W. Marciano and H. Pagels, Phys. Rep. 36C, 137 (1978); L-N. Chang and N-P. Chang, Phys. Rev. D29, 312 (1984).

26. L. S. Celenza and C. M. Shakin, Phys. Rev. D34, 1571 (1986); A. Le Yaouanc, L. Oliver, O. Pene, and J-C. Raynal, Phys. Rev. D29, 133 (1984).

27. D. Lurie, Particles and Fields (Interscience, New York, 1968) page 433.

Figure Captions

Fig. 1. The perturbation series for the Coulomb Green's function is portrayed in (a). Solid lines are $1/p^2$ propagators and wavy lines are gluons. The perturbation series is re-summed in (b); the dashed line is the dressed propagator function D(p). The nature of the integral equation for the vertex function is shown in (c).

Fig. 2. The order g^2 gluon counterterm, marked by a cross in (a), cancels four diagrams. Only the diagram with the singular Coulomb propagator indicated by a double dashed line survives the infrared limit. Quark lines are solid. The quark counterterm equation is shown in (b).

Fig. 3. Numerical solutions of the fundamental equations of the mean field model are shown as functions of p. The coupling constant in the equation for F(p) is multiplied by 0.9 to avoid complex solutions at $p = 0$. The logarithms of both D(p) and F(p) are plotted in (a), and the gluon propagator function A(p) is given in (b). F(p) is constrained by $F(1) = 1$. The quark counterterm functions $G_1(p)$ and $G_2(p)$ are zero on the scale of (b), except for the highest momenta. The momentum p is measured in units of $A(0)^{-1}$.

Fig. 4. A representative multi-gluon interaction arising from $V_1(A)$ is shown in (a); vertices are taken to be points. The broken wavy line represents a fictitious gluon line. Order g^4 and g^6 correction to the gluon self energy appear in (b). A large class of planar diagrams sum in (c) to produce vertex corrections for the fictitious gluon.

Fig. 5. The anomalous interaction $V_2(A)$ produces seven diagrams which contribute to the gluon self energy. All have two fictitious gluons (dashed wavy), two D(p) propagators (dashed), and a single F(p) (double dashed). Vertex effects are suppressed. The first three are produced by the first term in (A3); the second three dome from the second term in (A3). The final diagram arises from the third term.

Fig. 6. The one gluon term in the operator product expansion of $D_{ab}(\vec{p},\vec{k};x)$ is given in (a) and the two gluon term in (b). In the absence of vertex corrections, $F_{ab}(\vec{p},\vec{k};x)$ has the operator expansion in (c).

Fig. 7. The first correction to a point vertex on a D-type Coulomb line is shown in (a), while (b) suggests the complexity of higher order vertices. Substitution of (a) into the equation for D(p) requires evaluation of (c). A second correction to D(k) is produced by (d). The double dashed line is an F(p) propagator. Diagrams (e), (f), (g), and (h) are the g^4 corrections to the gluon self energy associated with the two gluon term in the operator expansion of $F_{ab}(\vec{p},\vec{k};x)$. Each diagram shows one of three positions for the singular Coulomb line. Diagram (i) is the quark loop term in the gluon propagator, and (j) and (k) show gluon and quark loop corrections to the Coulomb interaction between gluons.

Fig. 8. Two D-type Coulomb lines interact by gluon exchange. The ladder diagrams and crossed ladder diagrams are separately summed to produce Bethe-Salpeter-like wave functions.

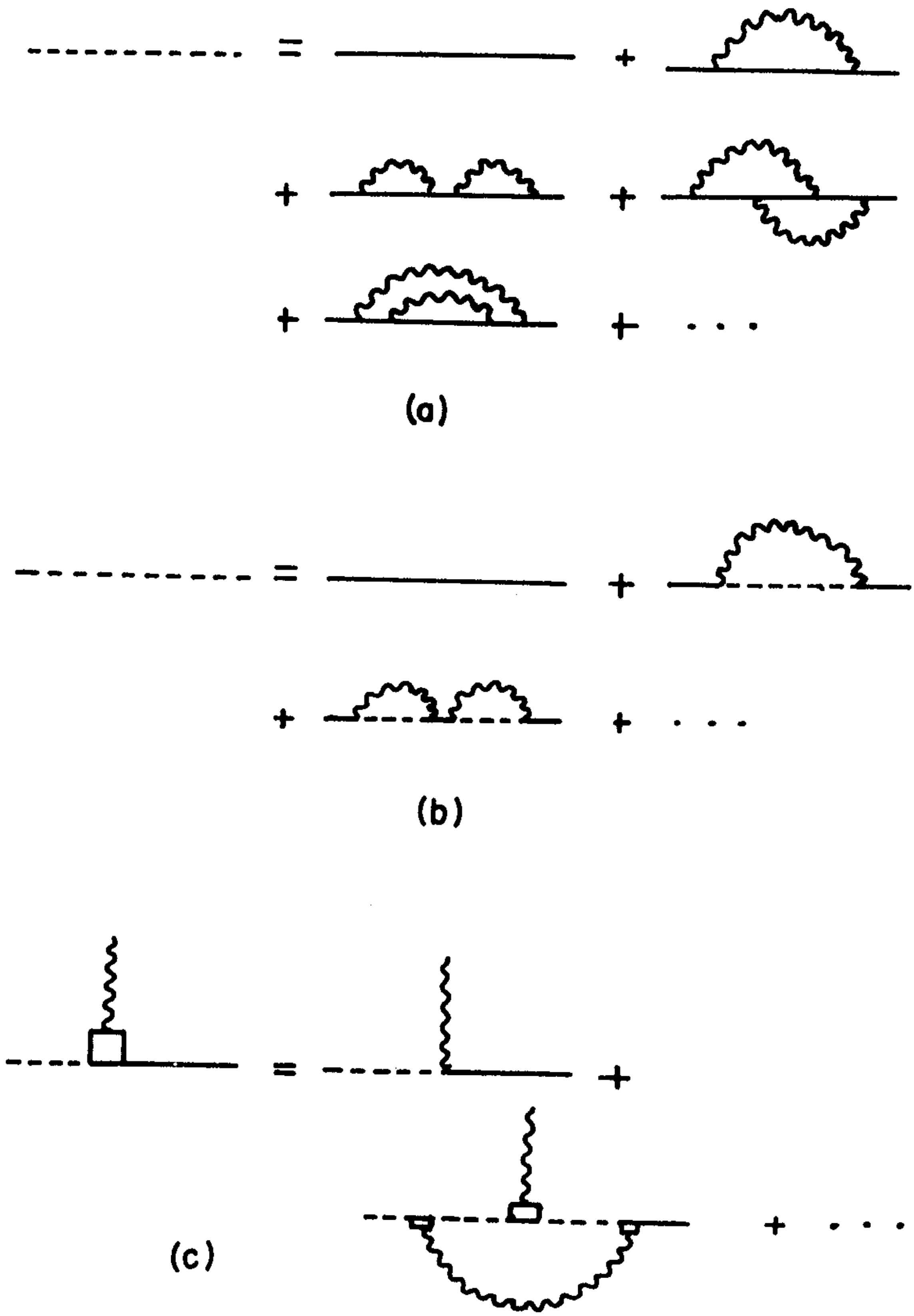


Figure 1

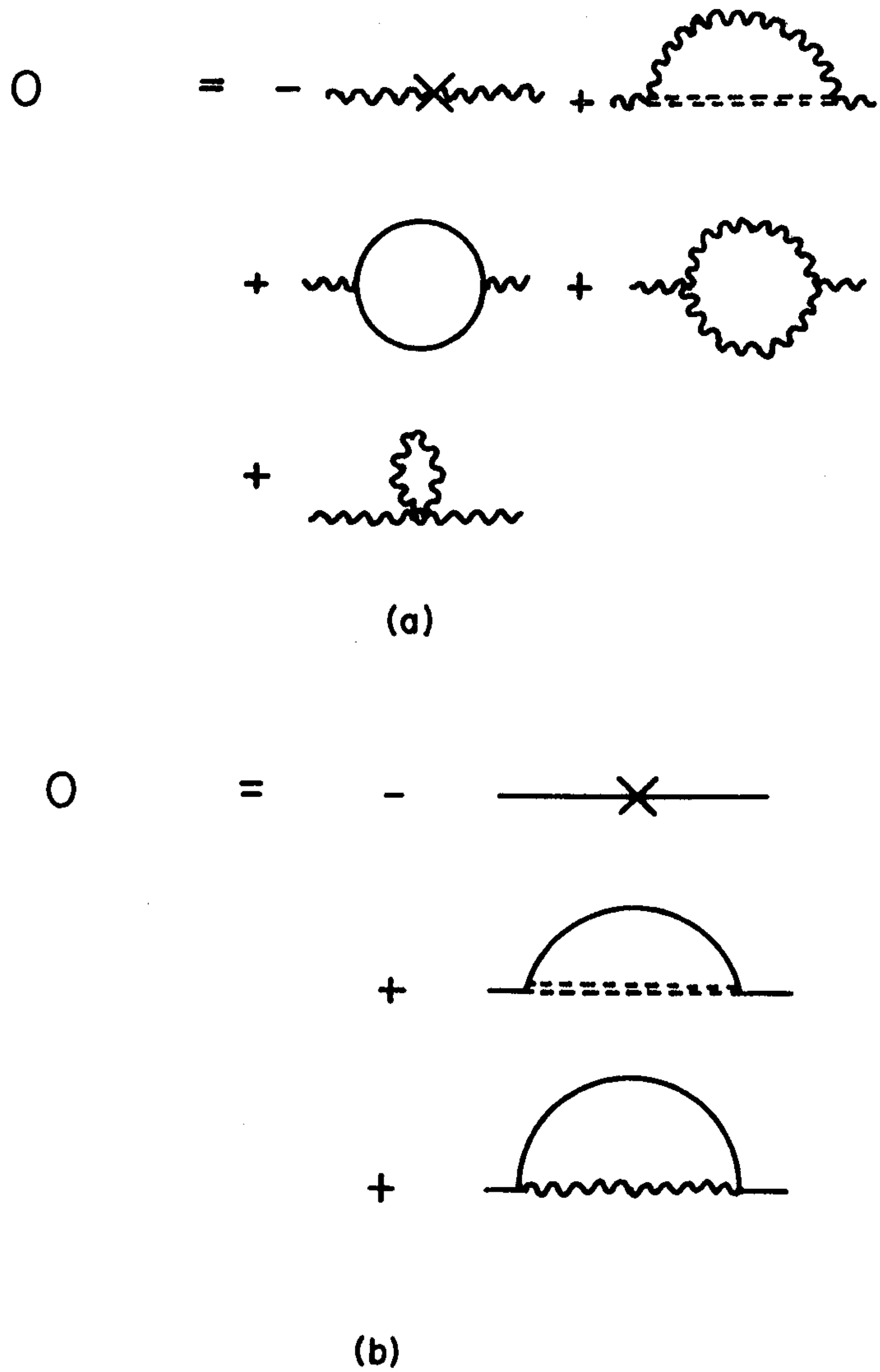
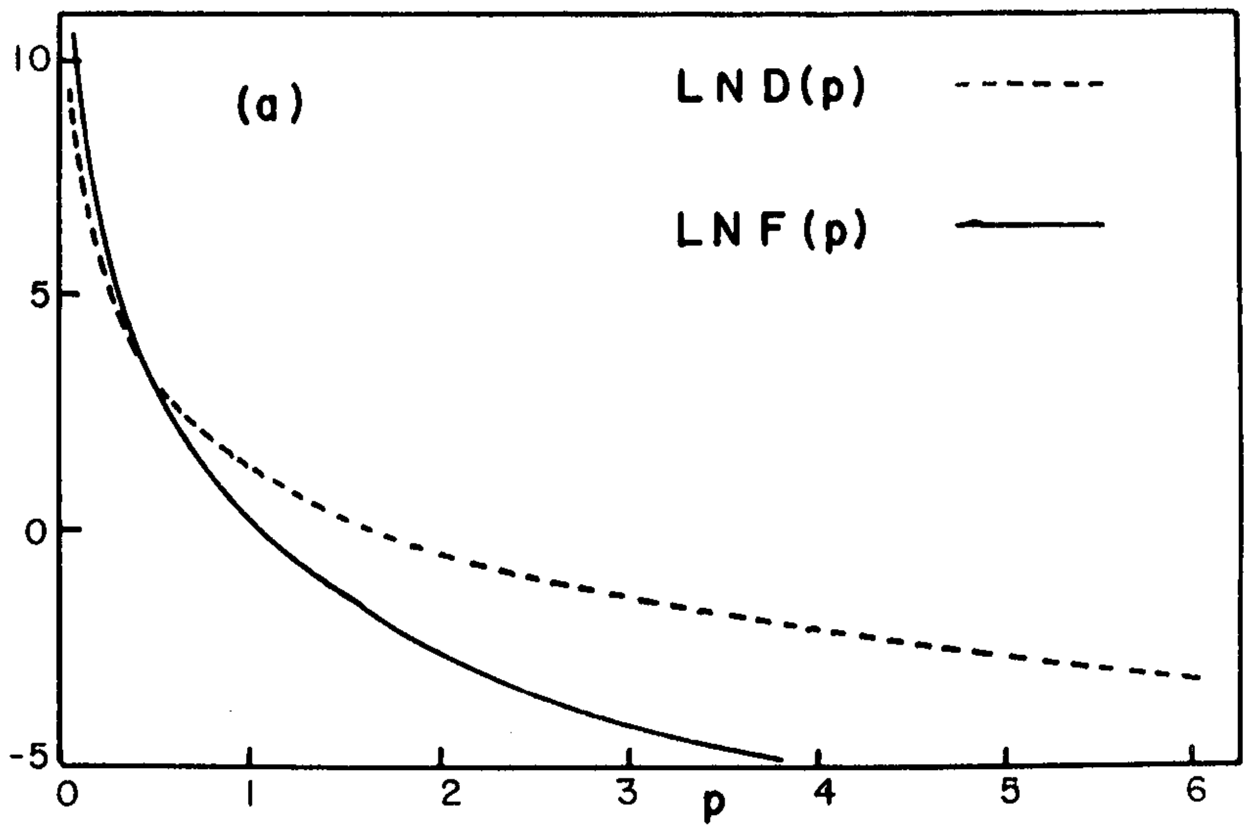
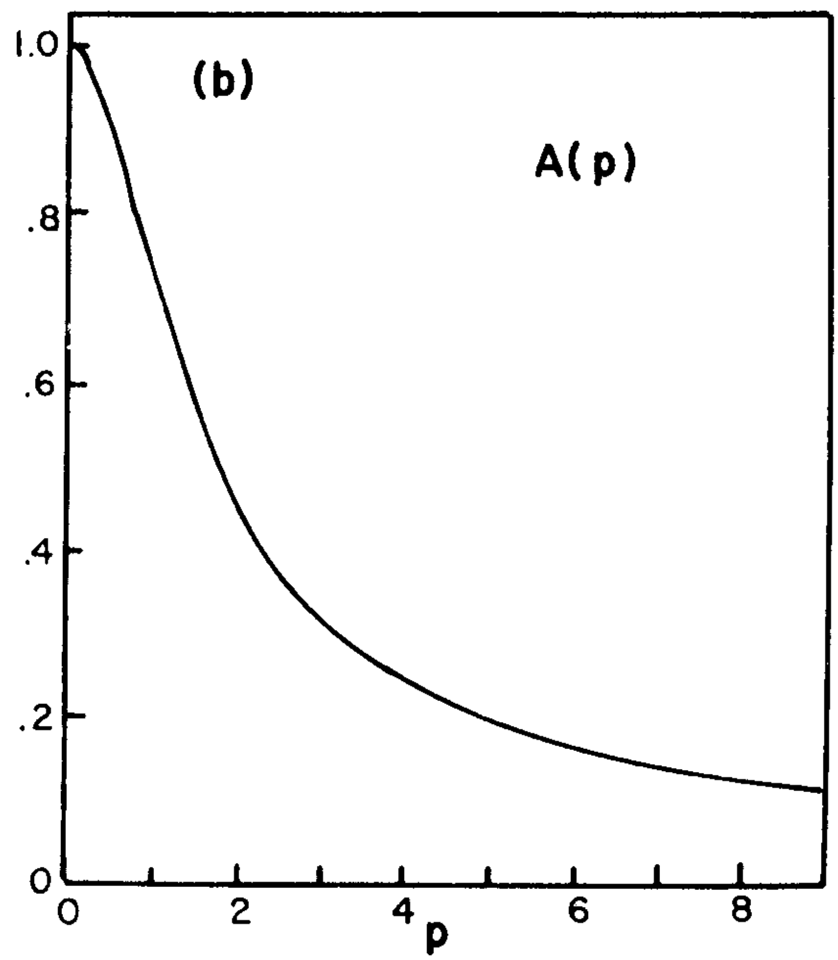


Figure 2

Figure 3





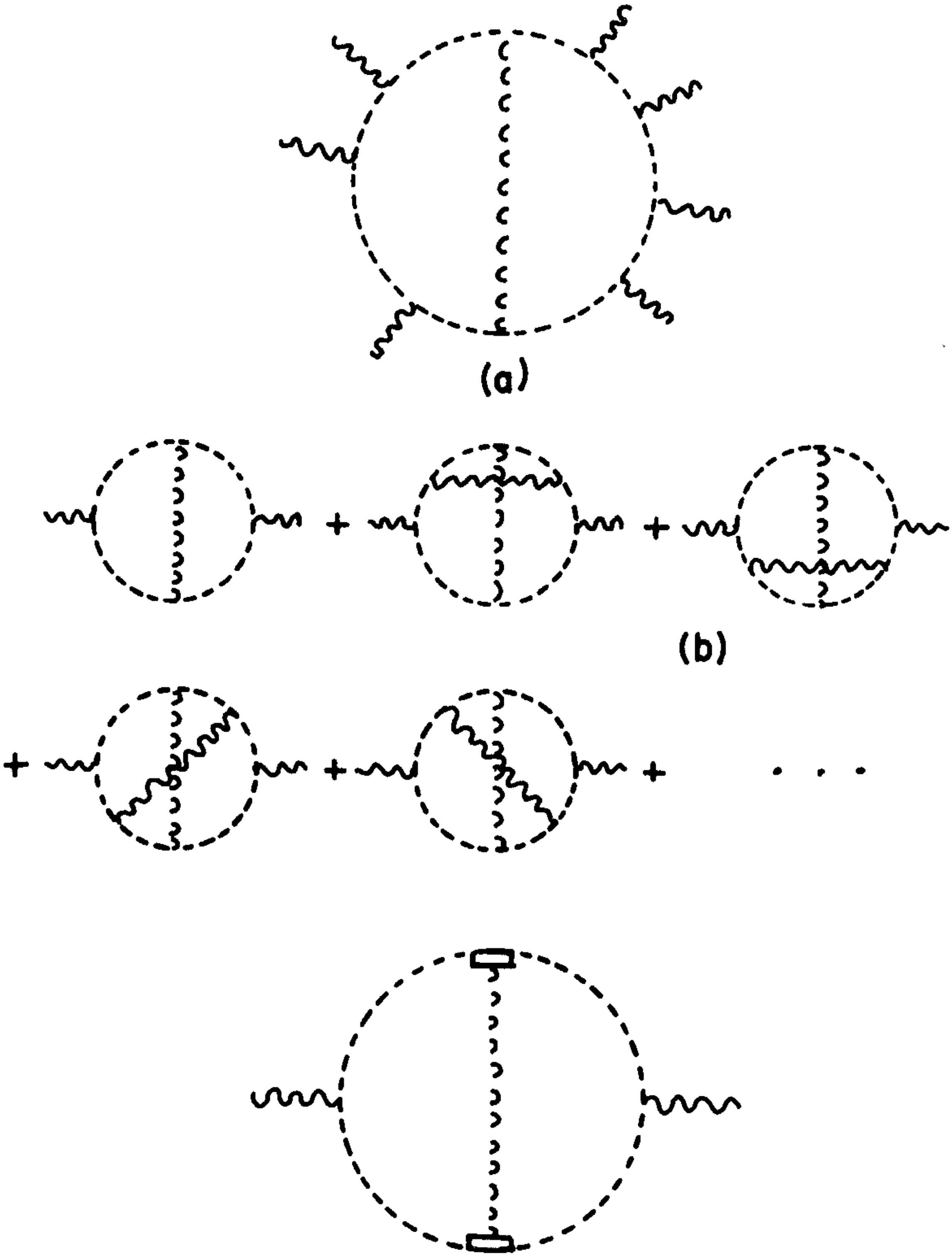


Figure 4

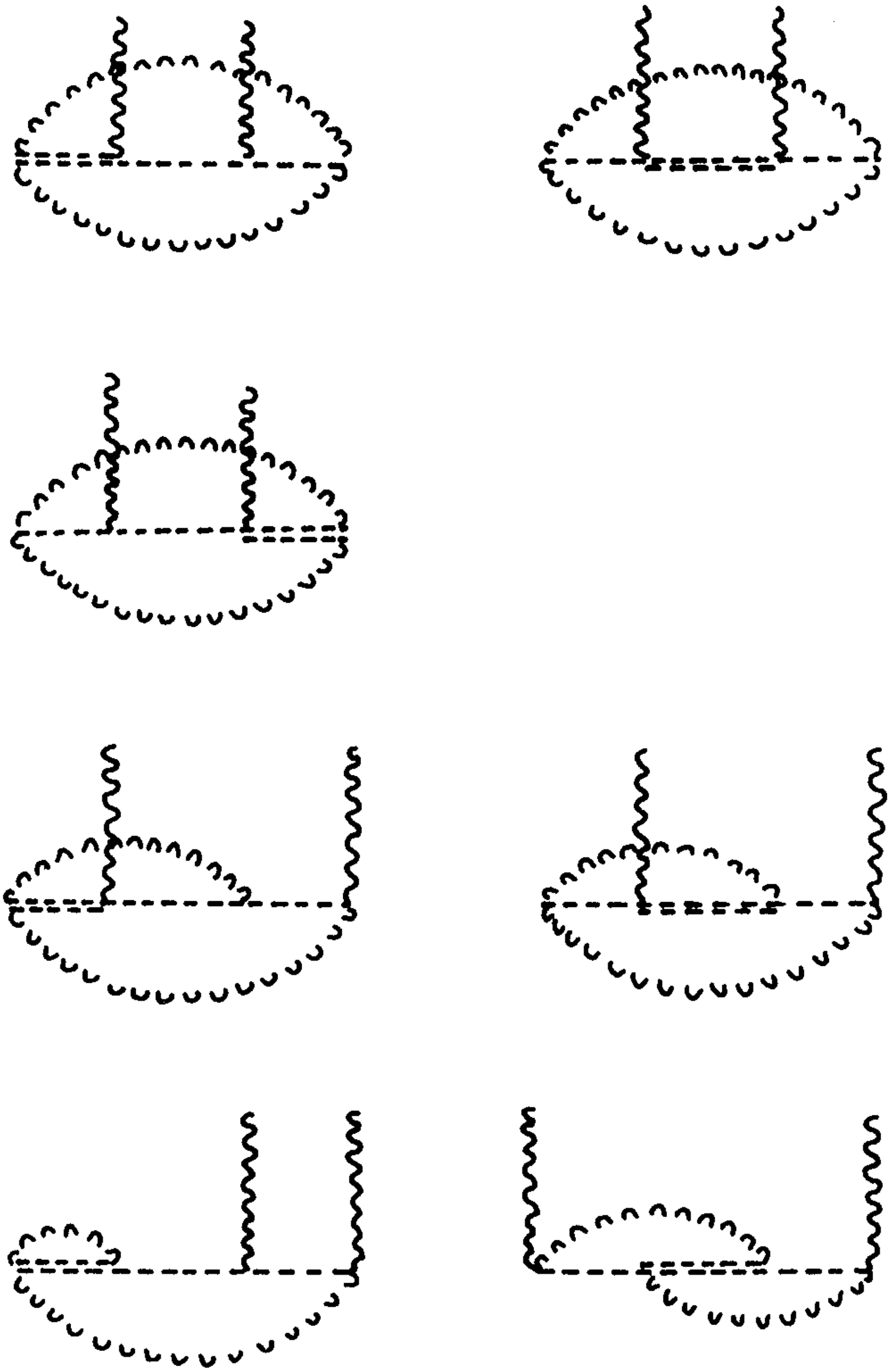


Figure 5

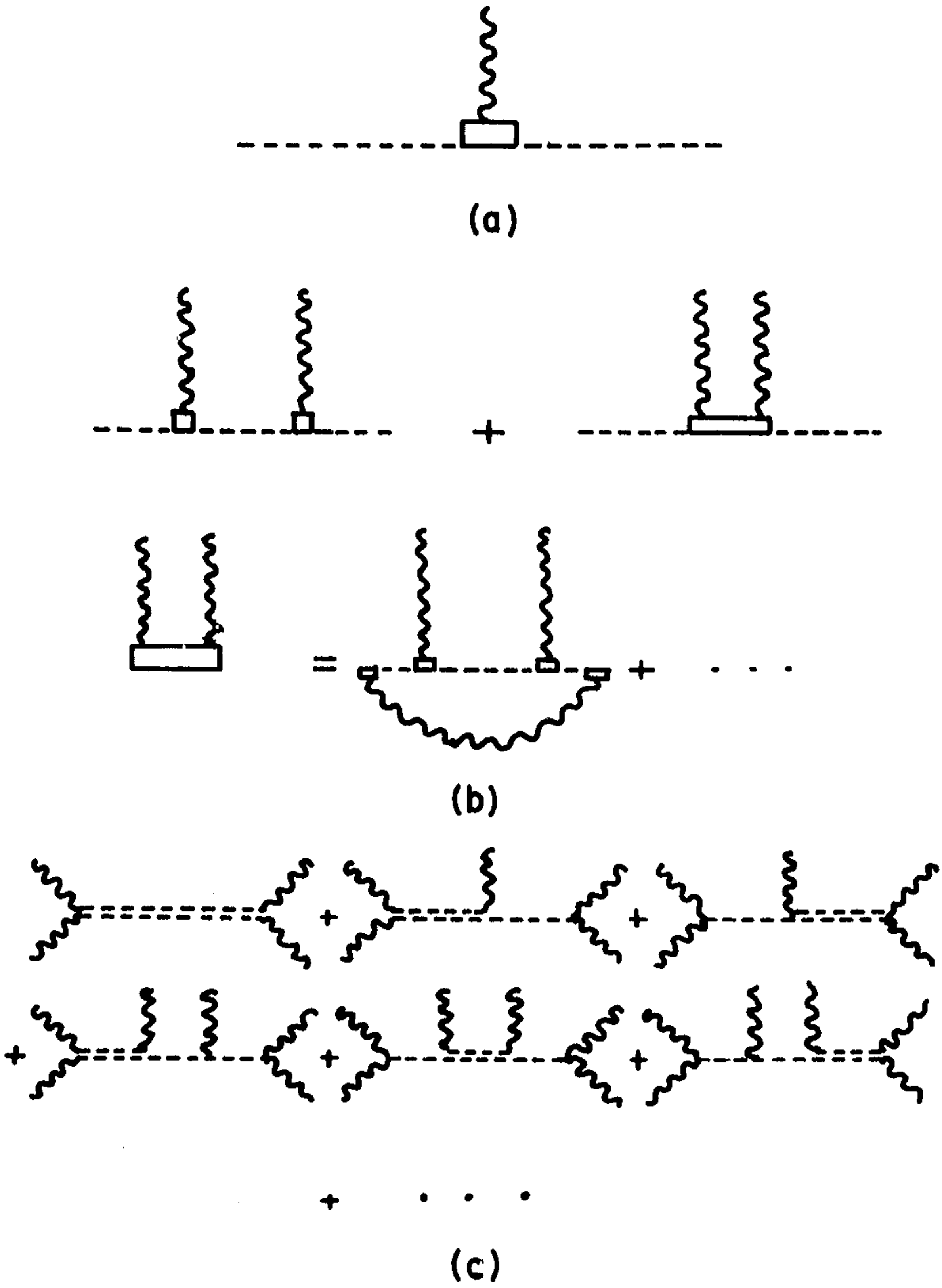


Figure 6

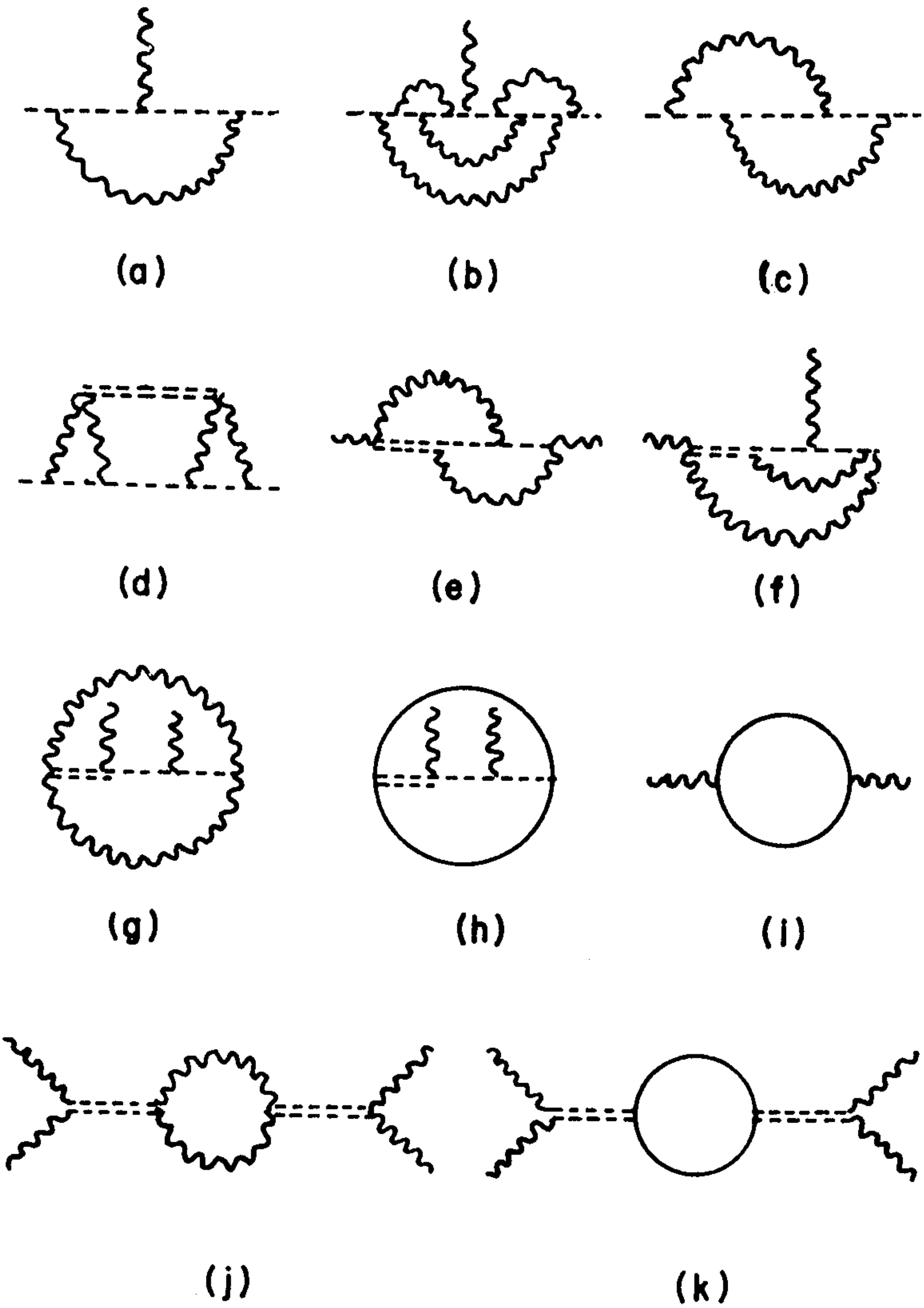


Figure 7

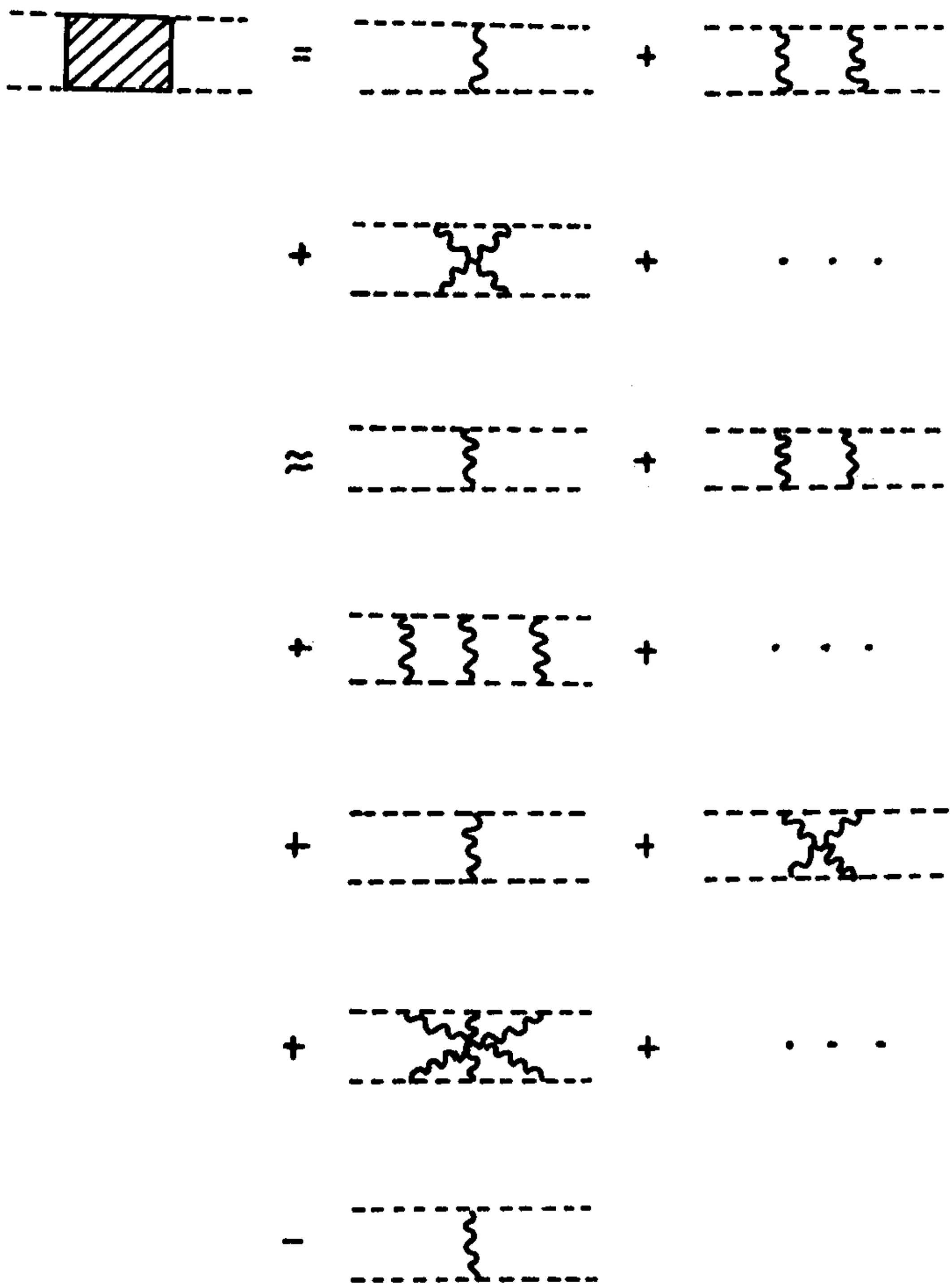


Figure 8